Triangulated quotient categories

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Abstract

A notion of mutation of subcategories in a right triangulated category is defined in this paper. When $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}-$mutation pair in a right triangulated category $\mathcal{C}$, the quotient category $\mathcal{Z}/\mathcal{D}$ carries naturally a right triangulated structure. Moreover, if the right triangulated category satisfies some reasonable conditions, then the right triangulated quotient category $\mathcal{Z}/\mathcal{D}$ becomes a triangulated category. When $\mathcal{C}$ is triangulated, our result unifies the constructions of the quotient triangulated categories by Iyama-Yoshino and by Jørgensen respectively.

Key words. Right triangulated category, mutation, quotient triangulated category.

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1 Introduction

Triangulated categories are important structure in algebra and geometry. There are two major ways to produce triangulated categories: forming homotopy or derived categories of abelian categories; and forming the stable category of Frobenius categories [H] [BR] [ASS] [N].

Among the surprises produced by the recent study on cluster algebras and cluster tilting theory is the possibility to define the notion of mutation in a triangulated category by Iyama-Yoshino [IY], which is a generalization of mutation of cluster tilting objects in cluster categories [BMRRT][KR][KZ]. The latter models the mutation of clusters of acyclic cluster algebras [FZ, K1, K2]. As one of main results in [IY], Iyama and Yoshino proved that if $\mathcal{D} \subseteq \mathcal{Z}$ are subcategories of a triangulated category $\mathcal{C}$, and if $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}-$mutation pair, where $\mathcal{D}$ is rigid, i.e. $Ext^1(\mathcal{D}, \mathcal{D}) = 0$, then the quotient category $\mathcal{Z}/\mathcal{D}$ is a triangulated category. Soon later, Jørgensen [J] gave a similar construction of triangulated category by quotient category in another manner. He proved that if $\mathcal{X}$ is a functorially finite subcategory of a triangulated category $\mathcal{C}$ with Auslander-Reiten translate τ, and if $\mathcal{X}$ satisfies the equation $\tau\mathcal{X} = \mathcal{X}$, then the quotient category $\mathcal{C}/\mathcal{X}$ is a triangulated category. Recently the authors define the mutation of torsion pairs in a triangulated category and give its geometric interpretation in [ZZ].

The aim of the paper is to unify these two constructions of the quotient triangulated categories by Iyama-Yoshino in [IY] and by Jørgensen in [J] respectively. We define the notion of $\mathcal{D}-$mutation without the assumption that $\mathcal{D}$ is rigid (compare [IY]). This

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generalizes the notion of $\mathcal{D}-$mutation defined in [IY] where $\mathcal{D}$ is assumed rigid. If $\mathcal{D}$ satisfies the condition of Theorem 2.3 in [J], i.e. $\mathcal{D}$ is a functorially finite subcategory of a triangulated category $\mathcal{C}$ which satisfies the equation $\tau\mathcal{X} = \mathcal{X}$, then $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{D}-$mutation in our sense. Finally we prove that if $\mathcal{D} \subseteq \mathcal{Z}$ are subcategories of a triangulated category $\mathcal{C}$, and $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}-$mutation pair, then the quotient category $\mathcal{Z}/\mathcal{D}$ is a triangulated category.

Actually, our setting is right triangulated categories which were defined and studied by Beligiannis, Assem and N.Marmaridis in [AB], [ABM].
The paper is organized as follows: In Section 2, we recall the definition of right triangulated category from [ABM], and define the notion of $\mathcal{D}-$mutation pair in it. We give some basic properties of right triangulated category and of its quotient categories which are needed in the proof of our main theorem. In Section 3, we state and prove the main results of this paper.

2 Right triangulated category

Throughout the paper, all the subcategories of a category are full subcategories and closed under isomorphisms. We recall some basics on right triangulated categories from [AB], [ABM].

Definition 2.1. *Let $\mathcal{C}$ be an additive category and T an additive endofunctor of $\mathcal{C}$. A sextuple (A, B, C, f, g, h) in $\mathcal{C}$ is given by objects $A, B, C \in \mathcal{C}$ and morphisms $f : A \to B$, $g : B \to C$ and $h : C \to TA$. A more suggestive notation of sextuple is*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA.$$

A morphism from sextuples (A, B, C, f, g, h) to (A', B', C', f', g', h') is a triple (a, b, c) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle Ta} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA'. \end{array}$$

If in addition a, b and c are isomorphisms in $\mathcal{C}$, the morphism is then called an isomorphism of sextuples.

A class $\sum$ of sextuples in $\mathcal{C}$ is called a right triangulation of $\mathcal{C}$ if the following conditions $TR(0) - TR(5)$ are satisfied. The elements of $\sum$ are then called right triangles in $\mathcal{C}$, and the tripe $(\mathcal{C}, T, \sum)$ is called a right triangulated category, or simply $\mathcal{C}$ is called a right triangulated category. The functor T is called the shift functor of the right triangulated category $\mathcal{C}$.
Thus if $T : \mathcal{C} \to \mathcal{C}$ is an equivalence, the right triangulated category $\mathcal{C}$ is a triangulated category. In this case, right triangles in $\mathcal{C}$ are called triangles [H].

$TR(0)$. $\sum$ is closed under isomorphisms.
$TR(1)$. For any $A \in \mathcal{C}$,

$$0 \stackrel{0}{\to} A \stackrel{1}{\to} A \stackrel{0}{\to} 0$$

is a right triangle.
$TR(2)$. Any morphism $f : A \to B$ in $\mathcal{C}$ can be extended to a right triangle

$$A \stackrel{f}{\to} B \stackrel{g}{\to} C \stackrel{h}{\to} TA.$$

$TR(3)$. If

$$A \stackrel{f}{\to} B \stackrel{g}{\to} C \stackrel{h}{\to} TA$$

is a right triangle, then

$$B \stackrel{g}{\to} C \stackrel{h}{\to} TA \stackrel{-Tf}{\to} TB$$

is a right triangle.
$TR(4)$. Given a commutative diagram where the rows are right triangles as follow:

$$\begin{array}{ccccccc} A & \stackrel{f}{\to} & B & \stackrel{g}{\to} & C & \stackrel{h}{\to} & TA \\ \downarrow a & & \downarrow b & & & & \downarrow Ta \\ A' & \stackrel{f'}{\to} & B' & \stackrel{g'}{\to} & C' & \stackrel{h'}{\to} & TA', \end{array}$$

there exists a morphism (a, b, c) from the first right triangle to the second.
$TR(5)$. (Octahedral axiom) Consider right triangles $X \stackrel{a}{\to} Y \stackrel{b}{\to} Z \stackrel{c}{\to} TX$, $Y \stackrel{d}{\to} U \stackrel{e}{\to} V \stackrel{f}{\to} TY$ and $X \stackrel{da}{\to} U \stackrel{g}{\to} W \stackrel{h}{\to} TX$. Then there exist morphisms $l : Z \to W$ and $i : W \to V$ such that the following diagrams commute and the third column in first diagram is a right triangle.

$$\begin{array}{ccccccc} X & \stackrel{a}{\to} & Y & \stackrel{b}{\to} & Z & \stackrel{c}{\to} & TX \\ \| & & \downarrow d & & \downarrow l & & \| \\ X & \stackrel{da}{\to} & U & \stackrel{g}{\to} & W & \stackrel{h}{\to} & TX \\ & & \downarrow e & & \downarrow i & & \\ & & V & = & V & & \\ & & \downarrow f & & \downarrow & & \\ & & TY & \stackrel{Tb}{\to} & TZ & & \end{array}$$

$$\begin{array}{ccccccc} X & \xrightarrow{da} & U & \xrightarrow{g} & W & \xrightarrow{h} & TX \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle Ta} \\ Y & \xrightarrow{d} & U & \xrightarrow{e} & V & \xrightarrow{f} & TY \end{array}$$

Proposition 2.2. *Let $\mathcal{C}$ be a right triangulated category, $A \stackrel{f}{\to} B \stackrel{g}{\to} C \stackrel{h}{\to} TA$ be a right triangle and E an object in $\mathcal{C}$. Then we have the following long exact sequence:*

$$Hom_{\mathcal{C}}(A, E) \stackrel{\circ f}{\leftarrow} Hom_{\mathcal{C}}(B, E) \stackrel{\circ g}{\leftarrow} Hom_{\mathcal{C}}(C, E) \stackrel{\circ h}{\leftarrow} Hom_{\mathcal{C}}(TA, E) \leftarrow \cdots$$

Proof. It is enough to show that

$$Hom(A,E) \overset{\circ f}{\leftarrow} Hom(B,E) \overset{\circ g}{\leftarrow} Hom(C,E)$$

is exact. By $TR(1)$,$TR(3)$ and $TR(4)$, we obtain that the following commutative diagrams of right triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & TA \\ \downarrow 1 & & \downarrow f & & \downarrow 0 & & \downarrow 1 \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \end{array}$$

Hence $gf = 0$, i.e. $im(\circ g) \subseteq ker(\circ f)$. If we have $if = 0$, $i \in Hom_{\mathcal{C}}(B,E)$, by $TR(1)$, we obtain that the following commutative diagrams of right triangles:

$$\begin{array}{ccccccc} A & \overset{f}{\to} & B & \overset{g}{\to} & C & \overset{h}{\to} & TA \\ \downarrow 0 & & \downarrow i & & & & \downarrow 0 \\ 0 & \overset{0}{\to} & E & \overset{1}{\to} & E & \overset{0}{\to} & 0 \end{array}$$

By $TR(4)$, there exists $j : C \to E$ such that $i = jg$, i.e. $ker(\circ f) \subseteq im(\circ g)$. Then $ker(\circ f) = im(\circ g)$. $\square$

Definition 2.3. *A subcategory $\mathcal{Z}$ of a right triangulated category $\mathcal{C}$ is called extension-closed if for any right triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} TA$ with $A, C \in \mathcal{Z}$, then we get $B \in \mathcal{Z}$.*

Definition 2.4. *Let $\mathcal{D}$ be a subcategory of a right triangulated category $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is called $\mathcal{D}-$epic, if for any $D \in \mathcal{D}$, we have that*

$$Hom_{\mathcal{C}}(D,A) \overset{f\circ}{\to} Hom_{\mathcal{C}}(D,B) \to 0$$

is exact. Dually, a morphism $f : A \to B$ in $\mathcal{C}$ is called $\mathcal{D}-$monic, if for any $D \in \mathcal{D}$,

$$Hom_{\mathcal{C}}(B,D) \overset{\circ f}{\to} Hom_{\mathcal{C}}(A,D) \to 0$$

is exact.

Definition 2.5. *Let $\mathcal{D}$ be a subcategory of a right triangulated category $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is called a right $\mathcal{D}-$approximation of B if $A \in \mathcal{D}$ and f is a $\mathcal{D}-$epic. Dually, a morphism $f : A \to B$ in $\mathcal{C}$ is called a left $\mathcal{D}-$approximation of A if $B \in \mathcal{D}$ and f is a $\mathcal{D}-$monic.*

Now we assume that a right triangulated category $\mathcal{C}$ is Krull-Schmidt, i.e. any object is isomorphic to a finite direct sum of objects whose endomorphism rings are local. When we say that $\mathcal{D}$ is a subcategory of $\mathcal{C}$, we always mean that $\mathcal{D}$ is full and is closed under isomorphisms, direct sums and direct summands.

The notion of $\mathcal{D}-$mutation of subcategories was defined in [IY] for a rigid subcategory $\mathcal{D}$. This notion generalizes the mutation of cluster tilting objects in cluster categories[BMRRT] which was motivated by modeling the mutation of clusters of cluster algebras [FZ]. In the following we recall the notion of $\mathcal{D}-$mutation of subcategories from [IY], but we don't assume that $\mathcal{D}$ is rigid here.

Definition 2.6. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be subcategories of a right triangulated category $\mathcal{C}$, and assume $\mathcal{D} \subseteq \mathcal{X}$, and $\mathcal{D} \subseteq \mathcal{Y}$. The subcategory $\mu^{-1}(\mathcal{X}; \mathcal{D})$ is defined as the subcategory consisting of objects $Y \in \mathcal{C}$ such that $Y \in \mathcal{D}$ or there exists a right triangle*

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} TX,$$

where $X \in \mathcal{X}$, $D \in \mathcal{D}$, f is a left $\mathcal{D}-$approximation and g is a right $\mathcal{D}-$approximation. Dually, for $\mathcal{Y}$, the subcategory $\mu(\mathcal{Y}; \mathcal{D})$ is defined as the subcategory consisting of objects $X \in \mathcal{C}$ such that $X \in \mathcal{D}$ or there exists a right triangle

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} TX,$$

where $Y \in \mathcal{Y}$, $D \in \mathcal{D}$, f is a left $\mathcal{D}-$approximation and g is a right $\mathcal{D}-$approximation. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{C}$ is called a $\mathcal{D}-$mutation pair if $\mu^{-1}(\mathcal{X}; \mathcal{D}) = \mathcal{Y}$ and $\mu(\mathcal{Y}; \mathcal{D}) = \mathcal{X}$.

Definition 2.7. *Let $\mathcal{D}$ be a subcategory of a right triangulated category $\mathcal{C}$. We denote by $[\mathcal{D}](X, Y)$ the subgroup of $Hom_{\mathcal{C}}(X, Y)$ consisting of morphisms which factor through an object in $\mathcal{D}$. We say that $\mathcal{D}$ is factor-through-epic if for any morphism $f \in [T^n\mathcal{D}](TX, TY)$ with $n > 0$, there exists a morphism $f' \in [T^{n-1}\mathcal{D}](X, Y)$ such that $Tf' = f$.*

Remark 2.8. *The zero subcategory $\mathcal{D} = 0$ is factor-through-epic. Another typical case is that: when $\mathcal{C}$ is a triangulated category, then any subcategory D is factor-through-epic.*

Lemma 2.9. *For any two objects A, B of a right triangulated category $\mathcal{C}$,*

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} TA$$

is a right triangle, where i_A is a section and p_B is a retraction.

Proof. By $TR(2)$, we can extend the section $A \xrightarrow{i_A} A \oplus B$ into a right triangle:

$$(*): \quad A \xrightarrow{i_A} A \oplus B \xrightarrow{p} C \xrightarrow{c} TA.$$

By $TR(1)$, $TR(3)$ and $TR(4)$, the following diagram where the rows are right triangles commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{i_A} & A \oplus B & \xrightarrow{p} & C & \xrightarrow{c} & TA \\ \downarrow 1 & & \downarrow p_A & & \downarrow 0 & & \downarrow 1 \\ A & \xrightarrow{1} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & TA \end{array}$$

It follows that $c = 0$. Since $p_B \circ i_A = 0$, by Prop 2.2, there exists a morphism $f : C \to B$ such that $p_B = fp$. From the following diagram:

$$\begin{array}{ccccccc} & & B & \xleftarrow{f} & C & & \\ & & \downarrow i_B & & \| & & \\ A & \xrightarrow{i_A} & A \oplus B & \xrightarrow{p} & C & \xrightarrow{0} & TA \\ & & \downarrow p_B & & \| & & \\ & & B & \xleftarrow{f} & C & & \end{array},$$

we know that $(1_C - p\circ i_B \circ f)\circ p = p\circ(1_{A\oplus B} - i_B\circ f\circ p) = p\circ i_A\circ p_A = 0$. By Prop 2.2, $1_C - p\circ i_B\circ f$ factors through $C \xrightarrow{0} TA$, hence $1_C - p\circ i_B\circ f = 0$, and $1_C = p\circ i_B\circ f$. We also have $f\circ(p\circ i_B) = (f\circ p)\circ i_B = 1_B$. Then f is an isomorphism. Thus the sextuple $A \xrightarrow{i_A} A\oplus B \xrightarrow{p_B} B \xrightarrow{0} TA$ is isomorphic to the right triangle $(*)$ under $(1_A, 1_{A\oplus B}, f)$. Hence it is a right triangle. □

Assumption 2.10. *From now on to the end of the paper, we assume that $\mathcal{C}$ is a right triangulated category and satisfies: If $B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB$ is a right triangle, then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ is a right triangle.*

Lemma 2.11. *Let $\mathcal{C}$ be a right triangulated category satisfying Assumption 2.10. Then the shift functor T is faithful:*

Proof. By Lemma 2.9, for any objects A and B, there exists a right triangle:

$$B \xrightarrow{i_B} B\oplus TA \xrightarrow{p_{TA}} TA \xrightarrow{0} TB.$$

For any morphism $f : A \to B$, if $Tf = 0$, by the Assumption 2.10, we have the right triangle

$$A \xrightarrow{f} B \xrightarrow{i_B} B\oplus TA \xrightarrow{p_{TA}} TA.$$

By Proposition 2.2, $i_B f = 0$, but i_B is a monomorphism, thus $f = 0$.

Remark 2.12. *Triangulated categories satisfy the Assumption 2.10. There are right triangulated categories satisfying the assumption, see Example 4 in Section 3.5.*

Proposition 2.13. *Let (a, b, c) be a morphism of right triangles in a right triangulated category $\mathcal{C}$:*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle Ta} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA'. \end{array}$$

If a and b are isomorphisms, so is c.

Proof. By applying the cohomological functor $Hom_{\mathcal{C}}(-, X)$ to the commutative diagram above, we obtain the following commutative diagram which has exact rows:

$$\begin{array}{ccccccccc} Hom_{\mathcal{C}}(TB,X) & \longrightarrow & Hom_{\mathcal{C}}(TA,X) & \longrightarrow & Hom_{\mathcal{C}}(C,X) & \longrightarrow & Hom_{\mathcal{C}}(B,X) & \longrightarrow & Hom_{\mathcal{C}}(A,X) \\ \downarrow{\scriptstyle Hom_{\mathcal{C}}(Tb,X)} & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(Ta,X)} & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(c,X)} & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(b,X)} & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(a,X)} \\ Hom_{\mathcal{C}}(TB',X) & \longrightarrow & Hom_{\mathcal{C}}(TA',X) & \longrightarrow & Hom_{\mathcal{C}}(C',X) & \longrightarrow & Hom_{\mathcal{C}}(B',X) & \longrightarrow & Hom_{\mathcal{C}}(A',X) \end{array}$$

By Snake-Lemma, $Hom_{\mathcal{C}}(c, X) : Hom_{\mathcal{C}}(C, X) \to Hom_{\mathcal{C}}(C', X)$ is an isomorphism for any $X \in C$, hence $Hom_{\mathcal{C}}(c, -) : Hom_{\mathcal{C}}(C', -) \to Hom_{\mathcal{C}}(C, -)$ is a functorial isomorphism. By Yoneda Lemma, c is an isomorphism. □

3 Quotient categories of a right triangulated category

3.1 Basics on quotient categories

Definition 3.1. *Let $\mathcal{D} \subset \mathcal{Z}$ be subcategories of a category $\mathcal{C}$. We denote by $\mathcal{Z}/\mathcal{D}$ the category whose objects are objects of $\mathcal{Z}$ and whose morphisms are elements of $Hom_{\mathcal{Z}}(X,Y)/[\mathcal{D}](X,Y)$ for $X, Y \in \mathcal{Z}$. Such category is called the quotient category of $\mathcal{Z}$ by $\mathcal{D}$. For any morphism $f : X \to Y$ in $\mathcal{Z}$, we denote by $\overline{f}$ the image of f under the natural quotient functor $\mathcal{Z} \to \mathcal{Z}/\mathcal{D}$.*

Lemma 3.2. *Let $\mathcal{D} \subseteq \mathcal{Z}$ be subcategories of a right triangulated category $\mathcal{C}$ and $\mathcal{D}$ be factor-through-epic. Consider the following commutative diagram:*

$$\begin{array}{ccccccc}
M & \xrightarrow{\alpha} & D & \xrightarrow{\beta} & S & \xrightarrow{\gamma} & TM \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle Tx} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX,
\end{array}$$

where rows are right triangles, $D \in \mathcal{D}$ and g is a $\mathcal{D} - epic$. If $\overline{z} = 0$ in the quotient category $\mathcal{Z}/\mathcal{D}$, then $\overline{x} = 0$.

Proof. $\overline{z} = 0$ means that z factors through an object $D' \in \mathcal{D}$. Since g is a $\mathcal{D} - epic$, we have the following commutative diagram:

$$\begin{array}{ccccc}
D' & \xrightarrow{1} & D' & \xleftarrow{a} & S \\
\downarrow c & & \downarrow b & & \downarrow z \\
Y & \xrightarrow{g} & Z & \xrightarrow{1} & Z.
\end{array}$$

Hence $z = ba = gca$. Then $Tx \circ \gamma = hz = hgca = 0$. By Proposition 2.2, there exists $\nu : TD \to TX$ which makes the diagram

$$\begin{array}{ccccccc}
TM & \xrightarrow{-T\alpha} & TD & \xrightarrow{-T\beta} & TS & \xrightarrow{-T\gamma} & T^2M \\
\downarrow{\scriptstyle Tx} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle T^2x} \\
TX & \xrightarrow{1} & TX & \xrightarrow{0} & 0 & \xrightarrow{0} & T^2X
\end{array}$$

commutative. Since $\mathcal{D}$ is factor-through-epic, there exists $\nu' : D \to X$ such that $\nu = T\nu'$. Hence $Tx = -T\nu' T\alpha$, which forces $x = -\nu'\alpha$, hence $\overline{x} = 0$. □

Lemma 3.3. *Let $\mathcal{D}$ be a subcategory of a right triangulated category $\mathcal{C}$ which is factor-through-epic. Consider the right triangle:*

$$A' \xrightarrow{f'} D \xrightarrow{g'} C' \xrightarrow{h'} TA'$$

where $D \in \mathcal{D}$. If $h' \circ c = 0$, for $c : C \to C'$, then we can find $d \in Hom_{\mathcal{C}}(C, D)$ such that $c = g'd$.

Proof. Since $h' \circ c = 0$, there exists a commutative diagram:

$$\begin{array}{ccccccc} C & \xrightarrow{0} & 0 & \xrightarrow{0} & TC & \xrightarrow{-1} & TC \\ \downarrow c & & \downarrow 0 & & \downarrow d_1 & & \downarrow Tc \\ C' & \xrightarrow{h'} & TA' & \xrightarrow{-Tf'} & TD & \xrightarrow{-Tg'} & TC', \end{array}$$

where the rows are right triangles, d_1 exists by $(TR4)$. Since $\mathcal{D}$ is factor-through-epic, there is a morphism $d \in Hom_{\mathcal{C}}(C, D)$ such that $d_1 = Td$. Hence $Tc = Tg'Td = T(g'd)$. Then $c = g'd$ by Lemma 2.11. □

Lemma 3.4. *Let $\mathcal{D} \subset \mathcal{Z}$ be subcategories of a right triangulated category $\mathcal{C}$ and $\mathcal{D}$ be factor-through-epic. Consider the following commutative diagram:*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow Ta \\ A' & \xrightarrow{f'} & D & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA', \end{array}$$

where the rows are right triangles, $D \in \mathcal{D}$ and f a left $\mathcal{D}-$monic. If $\overline{a} = 0$ in the quotient category $\mathcal{Z}/\mathcal{D}$, then $\overline{c} = 0$.

Proof. By the condition $\overline{a} = 0$, we have that a factors through an object $D_1 \in \mathcal{D}$. Since f is a left $\mathcal{D}-$*monic*, we have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{a} & A' \\ \downarrow f & & \downarrow d_1 & & \downarrow 1 \\ B & \xrightarrow{d} & D_1 & \xrightarrow{b} & A'. \end{array}$$

Hence $a = bd_1 f$, and $h'c = Ta \circ h = T(bd_1 f)h = T(bd_1)T(f)h = 0$. By Lemma 3.3, c factors through D, so we get $\overline{c} = 0$. □

3.2 Right triangles on the quotient category

In this subsection, $\mathcal{C}$ denotes a right triangulated category satisfying Assumption 2.10. Assume that $\mathcal{D} \subseteq \mathcal{Z}$ are subcategories of $\mathcal{C}$, $\mathcal{D}$ is factor-through-epic, $\mathcal{Z}$ is extension-closed and satisfies $\mathcal{Z} = \mu(\mathcal{Z}; \mathcal{D})$. Then for any object $M \in \mathcal{Z}$, there exists a right triangle

$$M \overset{\alpha_M}{\to} D_M \overset{\beta_M}{\to} \sigma M \overset{\gamma_M}{\to} TM$$

with $\sigma M \in \mathcal{Z}$ and $D_M \in \mathcal{D}$, and moreover α_M is a left $\mathcal{D}-$approximation and β_M is a right $\mathcal{D}-$approximation.

Definition 3.5. *Let* $M \xrightarrow{\mu} N \xrightarrow{\gamma} P \xrightarrow{\varphi} TM$ *be a right triangle in* $\mathcal{C}$*, where* μ *is* $\mathcal{D}-$*monic and* $M, N, P \in \mathcal{Z}$*. Then there exists a commutative diagram where the rows are right triangles:*

$$\begin{array}{ccccccc} M & \xrightarrow{\mu} & N & \xrightarrow{\gamma} & P & \xrightarrow{\varphi} & TM \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 1} \\ M & \xrightarrow{\alpha_M} & D_M & \xrightarrow{\beta_M} & \sigma M & \xrightarrow{\gamma_M} & TM. \end{array}$$

Then we have the following sextuple in the quotient category $\mathcal{Z}/\mathcal{D}$*:*

$$(*): M \xrightarrow{\overline{\mu}} N \xrightarrow{\overline{\gamma}} P \xrightarrow{\overline{\pi}} \sigma M$$

We define the right triangles in $\mathcal{Z}/\mathcal{D}$ *as the sextuples which are isomorphic to* $(*)$*.*

Remark 3.6. *It is easy to prove that* σM *is unique up to isomorphism in the quotient category* $\mathcal{Z}/\mathcal{D}$*. So for any* $M \in \mathcal{Z}$*, we fix a right triangle*

$$M \xrightarrow{\alpha_M} D_M \xrightarrow{\beta_M} \sigma M \xrightarrow{\gamma_M} TM$$

In particular, for any $M \in \mathcal{D}$*, we fix a right triangle*

$$M \xrightarrow{1} M \xrightarrow{0} 0 \xrightarrow{0} TM.$$

For any morphism $\mu \in Hom_{\mathcal{Z}}(M, N)$, where $M, N \in \mathcal{Z}$, there exist g and μ' which make the following diagram commutative.

$$\begin{array}{ccccccc} M & \xrightarrow{\alpha_M} & D_M & \xrightarrow{\beta_M} & \sigma M & \xrightarrow{\gamma_M} & TM \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle T\mu} \\ N & \xrightarrow{\alpha_N} & D_N & \xrightarrow{\beta_N} & \sigma N & \xrightarrow{\gamma_N} & TN. \end{array}$$

We define an endofunctor $\sigma : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ as follows: $\sigma : M \mapsto \sigma(M)$, $\overline{\mu} \mapsto \overline{\mu'}$.

Proposition 3.7. $\sigma : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ *is an additive functor.*

Proof. One can easily check that σ satisfies the definition of the additive functor. We only prove that σ is well-defined. Now assume $\mu, \mu_1 \in Hom_{\mathcal{Z}}(M, N)$, $\overline{\mu} = \overline{\mu_1}$. Then we also have the commutative diagram where the rows are right triangles (compare to the commutative diagram before the proposition):

$$\begin{array}{ccccccc} M & \xrightarrow{\alpha_M} & D_M & \xrightarrow{\beta_M} & \sigma M & \xrightarrow{\gamma_M} & TM \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \mu_1'} & & \downarrow{\scriptstyle T\mu_1} \\ N & \xrightarrow{\alpha_N} & D_N & \xrightarrow{\beta_N} & \sigma N & \xrightarrow{\gamma_N} & TN. \end{array}$$

Since $\overline{\mu} - \overline{\mu_1} = 0$, by Lemma 3.4, we have that $\overline{\mu'} = \overline{\mu_1'}$ □

Lemma 3.8. *Assume that we have a commutative diagram where the rows are right triangles in $\mathcal{C}$:*

$$\begin{array}{ccccccc}
M & \xrightarrow{\mu} & N & \xrightarrow{\gamma} & P & \xrightarrow{\varphi} & TM \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\
M' & \xrightarrow{\mu'} & N' & \xrightarrow{\gamma'} & P' & \xrightarrow{\varphi'} & TM',
\end{array}$$

where $M, N, P, M', N', P' \in \mathcal{Z}$ and μ and μ' are $\mathcal{D}-$monic. Then we have the following commutative diagram in $\mathcal{Z}/\mathcal{D}$:

$$\begin{array}{ccccccc}
M & \xrightarrow{\overline{\mu}} & N & \xrightarrow{\overline{\gamma}} & P & \xrightarrow{\overline{\pi}} & \sigma M \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \overline{h}} & & \downarrow{\scriptstyle \sigma(\overline{f})} \\
M' & \xrightarrow{\overline{\mu'}} & N' & \xrightarrow{\overline{\gamma'}} & P' & \xrightarrow{\overline{\pi'}} & \sigma M'
\end{array}$$

Proof. Consider the following commutative diagrams where the rows are right triangles in $\mathcal{C}$, and $\overline{f'} = \sigma(\overline{f})$:

$$\begin{array}{ccccccc}
M' & \xrightarrow{\mu'} & N' & \xrightarrow{\gamma'} & P' & \xrightarrow{\varphi'} & TM' \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle n'} & & \downarrow{\scriptstyle \pi'} & & \downarrow \\
M' & \xrightarrow{\alpha_{M'}} & D_{M'} & \xrightarrow{\beta_{M'}} & \sigma M' & \xrightarrow{\gamma_{M'}} & TM'
\end{array}$$

and

$$\begin{array}{ccccccc}
M & \xrightarrow{\alpha_M} & D_M & \xrightarrow{\beta_M} & \sigma M & \xrightarrow{\gamma_M} & TM \\
\downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle Tf} \\
M' & \xrightarrow{\alpha_{M'}} & D_{M'} & \xrightarrow{\beta_{M'}} & \sigma M' & \xrightarrow{\gamma_{M'}} & TM'.
\end{array}$$

We have that $\gamma_{M'}(f'\pi - \pi'h) = Tf \circ \gamma_M \circ \pi - \varphi'h = Tf \circ \varphi - \varphi'h = 0$. It follows from Lemma 3.3 that $\overline{f'\pi} = \overline{\pi'h}$. Then the following diagram commutes

$$\begin{array}{ccccccc}
M & \xrightarrow{\overline{\mu}} & N & \xrightarrow{\overline{\gamma}} & P & \xrightarrow{\overline{\pi}} & \sigma M \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \overline{h}} & & \downarrow{\scriptstyle \sigma(\overline{f})} \\
M' & \xrightarrow{\overline{\mu'}} & N' & \xrightarrow{\overline{\gamma'}} & P' & \xrightarrow{\overline{\pi'}} & \sigma M'
\end{array}$$

□

3.3 Main theorem

Theorem 3.9. *Let $\mathcal{C}$ be a right triangulated category satisfying Assumption 2.10. Assume that $\mathcal{D} \subseteq \mathcal{Z}$ are subcategories of $\mathcal{C}$, $\mathcal{D}$ is factor-through-epic, $\mathcal{Z}$ is extension-closed and satisfies $\mathcal{Z} = \mu(\mathcal{Z}; \mathcal{D})$. Then the quotient category $\mathcal{Z}/\mathcal{D}$ forms a right triangulated category with the additive functor σ and the right triangles defined in Def 3.5.*

Proof. We will check that the right triangles in $\mathcal{Z}/\mathcal{D}$ defined in Definition 3.5 satisfies the axioms of right triangulated categories (see Definition 2.1). It follows from the definition of right triangles in $\mathcal{Z}/\mathcal{D}$ that $TR(0)$ is satisfied.
For $TR(1)$: The commutative diagram

$$\begin{array}{ccccccc} 0 & \xrightarrow{0} & M & \xrightarrow{1} & M & \xrightarrow{0} & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \end{array}$$

shows that $0 \xrightarrow{0} M \xrightarrow{\overline{1}} M \xrightarrow{0} 0$ is a right triangle.
For $TR(2)$: Let $\overline{\mu} : M \to N$ be any morphism in $\mathcal{Z}/\mathcal{D}$. We have a morphism $\mu' = \begin{pmatrix} \mu \\ \alpha_M \end{pmatrix}$:$M \to N \oplus D_M$ which is $\mathcal{D}-monic$, where $\alpha_M : D_M \to M$ is a left $\mathcal{D}-$approximation.
Suppose that $M \xrightarrow{\mu'} N \oplus D_M \to P' \to TM$ is a right triangle in $\mathcal{C}$ which contains the morphism μ' as a part. It follows from Lemma 2.9 and the octahedral axiom that we have the following commutative diagram where the first two rows and the second column are right triangles:

$$\begin{array}{ccccccc} M & \xrightarrow{\mu'} & N \oplus D_M & \xrightarrow{\gamma'} & P' & \to & TM \\ \| & & \downarrow p_{D_M} & & \downarrow & & \| \\ M & \xrightarrow{\alpha_M} & D_M & \to & \sigma M & \to & TM \\ & & \downarrow & & \downarrow & & \\ & & TN & = & TN & & \\ & & \downarrow -Ti_N & & \downarrow & & \\ & & TN \oplus TD_M & \xrightarrow{T\gamma'} & TP' & & \end{array}$$

It also follows the third column is a right triangle. Then by the Assumption 2.10, $N \xrightarrow{\gamma' i_N} P' \to \sigma M \to TN$ is a right triangle in $\mathcal{C}$, where $N, \sigma M \in \mathcal{Z}$. Since $\mathcal{Z}$ is extension-closed, we have that $P' \in \mathcal{Z}$. Thus there is a right triangle $M \xrightarrow{\overline{\mu}} N \to P' \to \sigma(M)$ in $\mathcal{Z}/\mathcal{D}$ which contains $\overline{\mu}$ as a part.
For $TR(3)$: Let $M \xrightarrow{\overline{\mu}} N \xrightarrow{\overline{\gamma}} P \xrightarrow{\overline{\pi}} \sigma M$ be a right triangle in $\mathcal{Z}/\mathcal{D}$. We assume that it is induced by the right triangle in $\mathcal{C}$: $M \xrightarrow{\mu} N \xrightarrow{\gamma} P \xrightarrow{\varphi} TM$. Considering the commutative diagram in Def 3.5, we have $T\alpha_M \circ \varphi = Tn \circ T\mu \circ \varphi = 0$. By the octahedral axiom and Lemma 2.9, we have the following commutative diagrams

$$\begin{array}{ccccccc} P & \xrightarrow{\varphi} & TM & \xrightarrow{-T\mu} & TN & \xrightarrow{-T\gamma} & TP \\ \| & & \downarrow & & \downarrow \gamma' & & \| \\ P & \xrightarrow{0} & TD_M & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & TD_M \oplus TP & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & TP \\ & & \downarrow & & \downarrow \pi' & & \\ & & T\sigma M & = & T\sigma M & & \\ & & \downarrow & & \downarrow & & \\ & & T^2 M & \xrightarrow{-T^2\mu} & T^2 N & & \end{array}$$

and

$$\begin{array}{ccccccc}
P & \xrightarrow{0} & TD_M & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & TD_M \oplus TP & \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} & TP \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle T\varphi} \\
TM & \xrightarrow{-T\alpha_M} & TD_M & \xrightarrow{-T\beta_M} & T\sigma M & \xrightarrow{-T\gamma_M} & T^2M.
\end{array}$$

Write γ' as $\gamma' = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$. Then $-T\gamma = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \gamma_2$. Since $\gamma_1 \in Hom_{\mathcal{C}}(TN, TD_M)$ and $\mathcal{D}$ is factor-through-epic, there exists $\gamma_1{}' \in Hom_{\mathcal{C}}(N, D_M)$ such that $T\gamma_1{}' = \gamma_1$. Write π' as $\pi' = \begin{pmatrix}\pi_1{}' & \pi_2{}'\end{pmatrix}$, then $\begin{pmatrix}\pi_1{}' & \pi_2{}'\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \pi_1{}' = -T\beta_M$, $-T\gamma_M \begin{pmatrix}\pi_1{}' & \pi_2{}'\end{pmatrix}$ $= \begin{pmatrix}-T\gamma_M \circ \pi_1{}' & -T\gamma_M \circ \pi_2{}'\end{pmatrix} = \begin{pmatrix}0 & -T\gamma_M \circ \pi_2{}'\end{pmatrix} = \begin{pmatrix}0 & T\varphi\end{pmatrix}$. Then $-T\gamma_M \circ \pi_2{}' = T\varphi$. Since $T\gamma_M \circ (\pi_2{}' + T\pi) = 0$, by Lemma 3.3, $\pi_2{}' + T\pi$ factors through TD_M, there exists π_2 such that $T\pi_2 = \pi_2{}' + T\pi$. By the commutative diagram before Proposition 3.7, we have the following commutative diagram where the rows are right triangles in $\mathcal{C}$:

$$\begin{array}{ccccccc}
\sigma M & \xrightarrow{-T\mu\circ\gamma_M} & TN & \xrightarrow{\gamma'} & TD_M \oplus TP & \xrightarrow{\pi'} & T\sigma M \\
\downarrow{\scriptstyle -\mu'} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle -T\mu'} \\
\sigma N & \xrightarrow{\gamma_N} & TN & \xrightarrow{-T\alpha_N} & TD_N & \xrightarrow{-T\beta_N} & T\sigma N
\end{array}$$

By the Assumption 2.10, we have the following commutative diagram where the rows are right triangles:

$$\begin{array}{ccccccc}
N & \xrightarrow{\begin{pmatrix}-\gamma_1{}'\\ \gamma\end{pmatrix}} & D_M \oplus P & \xrightarrow{(\beta_M, \pi-\pi_2)} & \sigma M & \xrightarrow{-T\mu\circ\gamma_M} & TN \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle -\mu'} & & \downarrow{\scriptstyle 1} \\
N & \xrightarrow{\alpha_N} & D_N & \xrightarrow{\beta_N} & \sigma N & \xrightarrow{\gamma_N} & TN,
\end{array}$$

where $\begin{pmatrix}-\gamma_1{}'\\ \gamma\end{pmatrix}$ $\mathcal{D}-monic$. Then $N \xrightarrow{\overline{\gamma}} P \xrightarrow{\overline{\pi}} \sigma M \xrightarrow{-\sigma(\overline{\mu})=-\overline{\mu'}} \sigma N$ is a right triangle.

For $TR(4)$: Suppose there is a commutative diagram where the rows are right triangles in $\mathcal{Z}/\mathcal{D}$:

$$(*) \quad \begin{array}{ccccccc}
A & \xrightarrow{\overline{f}} & B & \xrightarrow{\overline{g}} & C & \xrightarrow{\overline{h}} & \sigma A \\
\downarrow \overline{a} & & \downarrow \overline{b} & & & & \downarrow \sigma(\overline{a}) \\
A' & \xrightarrow{\overline{f}'} & B' & \xrightarrow{\overline{g}'} & C' & \xrightarrow{\overline{h}'} & \sigma A'.
\end{array}$$

By the definition of right triangles of $\mathcal{Z}/\mathcal{D}$ in Def 3.5, there exists a(not necessarily commutative)diagram where the rows are right triangles in $\mathcal{C}$:

$$(**) \quad \begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\
\downarrow a & & \downarrow b & & & & \downarrow Ta \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA'.
\end{array}$$

Since $\overline{f'a} = \overline{bf}$, the morphism $bf - f'a$ factors through an object $D \in \mathcal{D}$. Since f is $\mathcal{D}-monic$, we have the following commutative diagram

$$\begin{array}{ccccc} A & \stackrel{1}{\leftarrow} & A & \stackrel{1}{\rightarrow} & A \\ \downarrow f & & \downarrow \alpha_3 & & \downarrow \quad bf - f'a \\ B & \stackrel{\alpha_1}{\rightarrow} & D & \stackrel{\alpha_2}{\rightarrow} & B' \end{array}$$

Denote $\alpha_2\alpha_1$ by s. We have $sf = bf - f'a$. After replacing $b - s$ by b in the diagram $(**)$, whose images in $\mathcal{Z}/\mathcal{D}$ are the same, we can assume that $bf = f'a$. Therefore there exists a morphism $c : C \to C'$ such that (a, b, c) is a morphism of right triangles in $\mathcal{C}$. It follows from Lemma 3.8 that $(\overline{a}, \overline{b}, \overline{c})$ is a morphism of triangles in $(*)$.

For $TR(5)$: Let $X \stackrel{\overline{a}}{\to} Y \stackrel{\overline{b}}{\to} Z \stackrel{\overline{c}}{\to} TX$, $Y \stackrel{\overline{d}}{\to} U \stackrel{\overline{e}}{\to} V \stackrel{\overline{f}}{\to} TY$, and $X \stackrel{\overline{da}}{\to} U \stackrel{\overline{g}}{\to} W \stackrel{\overline{h}}{\to} TX$ be right triangles in $\mathcal{Z}/\mathcal{D}$. By the proof for $TR(2)$ above, we may assume that a and d are $\mathcal{D}-monic$. Then da is also $\mathcal{D}-monic$. Hence we have the following three right triangles in $\mathcal{C}$: $X \stackrel{a}{\to} Y \stackrel{b}{\to} Z \stackrel{c'}{\to} TX$, $Y \stackrel{d}{\to} U \stackrel{e}{\to} V \stackrel{f'}{\to} TY$, $X \stackrel{da}{\to} U \stackrel{g}{\to} W \stackrel{h'}{\to} TX$. By octahedral axiom, we have the following commutative diagrams in $\mathcal{C}$:

$$\begin{array}{ccccccc} X & \stackrel{a}{\to} & Y & \stackrel{b}{\to} & Z & \stackrel{c'}{\to} & TX \\ \| & & \downarrow d & & \downarrow l & & \| \\ X & \stackrel{da}{\to} & U & \stackrel{g}{\to} & W & \stackrel{h'}{\to} & TX \\ & & \downarrow e & & \downarrow i & & \\ & & V & = & V & & \\ & & \downarrow f' & & \downarrow & & \\ & & TY & \stackrel{Tb}{\to} & TZ & & \end{array}$$

and

$$\begin{array}{ccccccc} X & \xrightarrow{da} & U & \xrightarrow{g} & W & \xrightarrow{h'} & TX \\ \downarrow a & & \downarrow 1 & & \downarrow i & & \downarrow Ta \\ Y & \xrightarrow{d} & U & \xrightarrow{e} & V & \xrightarrow{f'} & TY \end{array}$$

We will show that l is $\mathcal{D}-monic$. Let $j : Z \to D$ be any morphism, where $D \in \mathcal{D}$. Since d is $\mathcal{D}-monic$, there exists a morphism $k : U \to D$ such that $kd = jb$. Then $kda = jba = 0$. By Prop 2.2, there exists a morphism $m : W \to D$ such that $mg = k$. Now we get $mlb = mgd = kd = jb$, thus $(ml - j)b = 0$. Then there exists a morphism $\alpha : TX \to D$ such that $ml - j = \alpha c' = \alpha h'l$. Then $j = (m - \alpha h')l$. Now by Lemma 3.8, we have the following commutative diagrams in $\mathcal{Z}/\mathcal{D}$ where the rows are right triangles:

$$\begin{array}{ccccccc} X & \stackrel{\overline{a}}{\to} & Y & \stackrel{\overline{b}}{\to} & Z & \stackrel{\overline{c}}{\to} & \sigma X \\ \| & & \downarrow \overline{d} & & \downarrow \overline{l} & & \| \\ X & \stackrel{\overline{da}}{\to} & U & \stackrel{\overline{g}}{\to} & W & \stackrel{\overline{h}}{\to} & \sigma X \\ & & \downarrow \overline{e} & & \downarrow \overline{i} & & \\ & & V & = & V & & \\ & & \downarrow \overline{f} & & \downarrow & & \\ & & \sigma Y & \stackrel{\sigma(\overline{b})}{\to} & \sigma Z & & \end{array}$$

and

$$\begin{array}{ccccccc} X & \xrightarrow{\overline{da}} & U & \xrightarrow{\overline{g}} & W & \xrightarrow{\overline{h}} & \sigma X \\ \downarrow{\scriptstyle \overline{a}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \overline{i}} & & \downarrow{\scriptstyle \sigma(\overline{a})} \\ Y & \xrightarrow{\overline{d}} & U & \xrightarrow{\overline{e}} & V & \xrightarrow{\overline{f}} & \sigma Y \end{array}$$

Therefore the quotient category $\mathcal{Z}/\mathcal{D}$ is a right triangulated category with shift functor σ. □

Corollary 3.10. *Assume that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{Z}$ satisfy the same conditions as in Theorem 3.9. Let $\mu \in Hom_{\mathcal{Z}}(M,N)$ be $\mathcal{D}-$monic, where $M, N \in \mathcal{Z}$. If $M \xrightarrow{\mu} N \xrightarrow{\gamma} P \xrightarrow{\varphi} TM$ is a right triangle in $\mathcal{C}$, then $P \in \mathcal{Z}$.*

Proof. According to the proof of Theorem 3.9, for any morphism $\mu \in Hom_{\mathcal{Z}}(M,N)$, there exists a right triangle $M \xrightarrow{\mu'} N \oplus D_M \to P' \to TM$ with $\mu' = \begin{pmatrix} \mu \\ \alpha_M \end{pmatrix}$ and $P' \in \mathcal{Z}$. Suppose $M \xrightarrow{\mu} N \xrightarrow{\gamma} P \xrightarrow{\varphi} TM$ is the right triangle in $\mathcal{C}$ containing μ as a part. We have the following commutative diagrams (compare the Definition 3.5), where the rows are right triangles in $\mathcal{C}$:

$$\begin{array}{ccccccc} M & \xrightarrow{\mu} & N & \to & P & \to & TM \\ \| & & \downarrow n' & & \downarrow g_1 & & \| \\ M & \xrightarrow{\mu'} & N \oplus D_M & \to & P' & \to & TM \\ \| & & \downarrow f & & \downarrow g_2 & & \| \\ M & \xrightarrow{\mu} & N & \to & P & \to & TM, \end{array}$$

where $n' = \begin{pmatrix} 1_N \\ n \end{pmatrix}$ and $f = \begin{pmatrix} 1_N & 0 \end{pmatrix}$. Since $fn' = 1_N$, by Prop 2.13, we have that $g_2 g_1$ is an isomorphism. Thus P is a direct summand of P'. Therefore $P \in \mathcal{Z}$. □

3.4 Triangulated structure on the quotient category $\mathcal{Z}/\mathcal{D}$

Theorem 3.11. *Let $\mathcal{C}$ be a right triangulated category satisfying Assumption 2.10. Assume that $\mathcal{D} \subseteq \mathcal{Z}$ are subcategories of $\mathcal{C}$, $\mathcal{D}$ is factor-through-epic, $\mathcal{Z}$ is extension-closed and $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}-$mutation pair. If the restriction of the shift functor T to $\mathcal{Z}$, $T|_{\mathcal{Z}} : \mathcal{Z} \to T\mathcal{Z}$ is full, then the right triangulated category $\mathcal{Z}/\mathcal{D}$ is a triangulated category.*

Proof. By Theorem 3.9, we only need to show that the shift functor σ in $\mathcal{Z}/\mathcal{D}$ is an equivalence. Since $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}-$mutation pair, for any object $X \in \mathcal{Z}$, we fix a right triangle in $\mathcal{C}$: $\omega X \xrightarrow{\alpha^X} D^X \xrightarrow{\beta^X} X \xrightarrow{\gamma^X} T\omega X$ with β^X being a right $\mathcal{D}-$approximation and α^X being a left $\mathcal{D}-$approximation. For any morphism $f : X \to Y$, there exist g and h such that (g, f, h) is a morphism of right triangles (i.e. the following diagram commutes):

$$\begin{array}{ccccccc} D^X & \xrightarrow{\beta^X} & X & \xrightarrow{\gamma^X} & T\omega X & \xrightarrow{-T\alpha^X} & TD^X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow \\ D^Y & \xrightarrow{\beta^Y} & Y & \xrightarrow{\gamma^Y} & T\omega Y & \xrightarrow{-T\alpha^Y} & TD^Y \end{array}$$

By the fullness of the functor $T|_{\mathcal{Z}}$, there exists $h' \in Hom_{\mathcal{C}}(\omega X, \omega X')$ such that $Th' = h$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
\omega X & \xrightarrow{\alpha^X} & D^X & \xrightarrow{\beta^X} & X & \xrightarrow{\gamma^X} & T\omega X \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Th'} \\
\omega Y & \xrightarrow{\alpha^Y} & D^Y & \xrightarrow{\beta^Y} & Y & \xrightarrow{\gamma^Y} & T\omega Y.
\end{array}$$

Now we define $\omega : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ as a functor which sends X to ωX, sends $\overline{f}$ to $\overline{h'}$. By using Lemma 3.2, one can prove that $\omega : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ is an additive functor in a way dual to the construction of σ (compare Proposiotion 3.7 and its proof). Then we have the following commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccccccc}
\omega X & \overset{\alpha^X}{\to} & D^X & \overset{\beta^X}{\to} & X & \overset{\gamma^X}{\to} & T\omega X \\
\| & & \downarrow & & \downarrow g_X & & \| \\
\omega X & \overset{\alpha_{\omega X}}{\to} & D_{\omega X} & \overset{\beta_{\omega X}}{\to} & \sigma\omega X & \overset{\gamma_{\omega X}}{\to} & T\omega X \\
\| & & \downarrow & & \downarrow g'_X & & \| \\
\omega X & \overset{\alpha^X}{\to} & D^X & \overset{\beta^X}{\to} & X & \overset{\gamma^X}{\to} & T\omega X.
\end{array}$$

It follows from the commutative diagram that $\gamma^X(1_X - g'g) = 0$ and $\gamma_{\omega X}(1_{\sigma\omega X} - gg') = 0$. By using Lemma 3.3, one can get that $\overline{g}_X \circ \overline{g_X}' = \overline{1}_{\sigma\omega X}$, $\overline{g_X}' \circ \overline{g}_X = \overline{1}_X$. Then $X \cong \sigma\omega X$ in $\mathcal{Z}/\mathcal{D}$. For any morphism $f : X \to Y$, we also have the following commutative diagram:

$$\begin{array}{ccccccc}
\omega X & \overset{\alpha_{\omega X}}{\to} & D_{\omega X} & \overset{\beta_{\omega X}}{\to} & \sigma\omega X & \overset{\gamma_{\omega X}}{\to} & T\omega X \\
\| & & \downarrow & & \downarrow g'_X & & \| \\
\omega X & \overset{\alpha^X}{\to} & D^X & \overset{\beta^X}{\to} & X & \overset{\gamma^X}{\to} & T\omega X \\
\downarrow h' & & \downarrow & & \downarrow f & & \downarrow \\
\omega Y & \overset{\alpha^Y}{\to} & D^Y & \overset{\beta^Y}{\to} & Y & \overset{\gamma^Y}{\to} & T\omega Y \\
\| & & \downarrow & & \downarrow g_Y & & \| \\
\omega Y & \overset{\alpha_{\omega Y}}{\to} & D_{\omega Y} & \overset{\beta_{\omega Y}}{\to} & \sigma\omega Y & \overset{\gamma_{\omega Y}}{\to} & T\omega Y.
\end{array}$$

Then we get the following commutative diagram

$$\begin{array}{ccccccc}
\omega X & \overset{\alpha_{\omega X}}{\to} & D_{\omega X} & \overset{\beta_{\omega X}}{\to} & \sigma\omega X & \overset{\gamma_{\omega X}}{\to} & T\omega X \\
\downarrow h' & & \downarrow & & \downarrow f'' & & \| \\
\omega Y & \overset{\alpha_{\omega Y}}{\to} & D_{\omega Y} & \overset{\beta_{\omega Y}}{\to} & \sigma\omega Y & \overset{\gamma_{\omega Y}}{\to} & T\omega Y.
\end{array}$$

where $f'' = g_Y \circ f \circ g'_X$. Then we have that $\overline{f''} = \sigma\overline{h}' = \sigma\omega\overline{f}$ in $\mathcal{Z}/\mathcal{D}$. We also have the following commutative diagram in $\mathcal{Z}/\mathcal{D}$:

$$\begin{array}{ccc}
X & \overset{\overline{g}_X}{\to} & \sigma\omega X \\
\downarrow \overline{f} & & \downarrow \overline{f''} \\
Y & \overset{\overline{g}_Y}{\to} & \sigma\omega Y
\end{array}$$

Since $\overline{g_X}$ and $\overline{g_Y}$ are isomorphisms in $\mathcal{Z}/\mathcal{D}$, $\sigma\omega$ is an equivalence. Dually one can show that $\omega\sigma$ is also an equivalence. This proves σ is an equivalence. $\square$

A direct application of Theorem 3.12 to the special case when $\mathcal{C}$ is a triangulated category is the following result.

Corollary 3.12. *Let $\mathcal{C}$ be a triangulated category, and $(\mathcal{Z}, \mathcal{Z})$ a $\mathcal{D}-$mutation pair. Then the quotient category $\mathcal{Z}/\mathcal{D}$ is a triangulated category.*

3.5 Examples

Example 1. *In [IY], for a rigid subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C}$, i.e. $\mathcal{D}$ satisfies $Hom_{\mathcal{C}}(\mathcal{D}, T\mathcal{D}) = 0$, Iyama and Yoshino defined $\mathcal{D}-$mutation pair in $\mathcal{C}$. It is easy to see that when $\mathcal{D}$ is rigid, and $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{D}-$ mutation pair, then $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{D}-$mutation pair in the sense of Definition 2.6. It follows from Corollary 3.12 that the quotient category $\mathcal{Z}/\mathcal{D}$ is a triangulated category, which is Theorem 4.2 in [IY].*

Example 2. *Let $\mathcal{C}$ be a triangulated category with shift functor T, $\mathcal{X} \subseteq \mathcal{C}$ is a functorially finite subcategory. Jørgensen proved in [J] that if $\mathcal{C}$ has Serre functor S which satisfies $T^{-1} \circ S\mathcal{X} = \mathcal{X}$, then in the triangle $M \to N \to P \to TP$, $M \to N$ is $\mathcal{X}-$monic if and only if $N \to P$ is $\mathcal{X}-$epic. So $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{X}-$ mutation pair defined in Definition 2.6. It follows from Corollary 3.12, the quotient category $\mathcal{C}/\mathcal{D}$ is a triangulated category, which is the sufficient part of Theorem 3.3 in [J].*

Example 3. *Let $\mathcal{C}$ be a $2-$Calabi-Yau triangulated category [KR,IY], and E an rigid object in $\mathcal{C}$, i.e. $Ext^1_{\mathcal{C}}(E, E) = 0$. The quotient category $\mathcal{C}/addE$ is a right triangulated category [ABM]. Let $\mathcal{X}$ be the subcategory of $\mathcal{C}$ consisting of objects X with $Hom_{\mathcal{C}}(E, X[1]) = 0$. It is easy to see that $(\mathcal{X}, \mathcal{X})$ is an addE-mutation pair in $\mathcal{C}$. Now passing to the right triangulated quotient category $\mathcal{C}/addE$, we get the quotient subcategory of $\mathcal{C}/addE$, denoted by $\mathcal{Z}$. Then $(\mathcal{Z}, \mathcal{Z})$ is a $0-$mutation pair in $\mathcal{C}/addE$, hence $\mathcal{Z}$ is a triangulated subcategory in $\mathcal{C}$ by Theorem 3.11.*

Example 4. *Let $(\mathcal{B}, \mathcal{S})$ be the exact category defined in [H]. Assume that $(\mathcal{B}, \mathcal{S})$ has enough $\mathcal{S}$-injectives, $\underline{\mathcal{B}}$ is the quotient category $\mathcal{B}/\mathcal{S}$ [H]. Then according to the theorem of Chapter 1.2 in [H], $\underline{\mathcal{B}}$ is a right triangulated category.*

Claim: *If all the $\mathcal{S}$-injectives are also $\mathcal{S}$-projectives (which means that the set of $S-$injectives is contained in the set of $S-$projectives) , then the shift functor in $\underline{\mathcal{B}}$ is fully and faithful. Moreover the right triangulated category $\underline{\mathcal{B}}$ satisfies the Assumption 2.10.*
Proof. *We will give a brief proof of this claim:*
Note that we will use the notations in [H], just changing "triangle" into "right triangle".
(i)For any morphism $f : X \to Y$ in $\mathcal{B}$, there exists a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & X & \xrightarrow{\mu(X)} & I(X) & \xrightarrow{\pi(X)} & TX & \xrightarrow{0} & 0 \\ & & \downarrow f & & \downarrow I(f) & & \downarrow f' & & \\ 0 & \xrightarrow{0} & Y & \xrightarrow{\mu(Y)} & I(Y) & \xrightarrow{\pi(Y)} & TY & \xrightarrow{0} & 0. \end{array}$$

Then $\underline{f'} = T\underline{f}$ in $\underline{\mathcal{B}}$ by the definition of T [H]. Suppose $\underline{f'} = 0$. Then f' factors through some $\mathcal{S}-$injective I. But I is also $\mathcal{S}-$projective, then f' factors through I(Y). Let $f' =$

$\pi(Y)\alpha$ with $\alpha : TX \to I(Y)$, then $\pi(Y)(I(f) - \alpha\pi(X)) = 0$, which means $I(f) - \alpha\pi(X)$ factors through Y. Let $I(f) - \alpha\pi(X) = \mu(Y)\beta$, where $\beta : I(X) \to Y$, then $\mu(Y)(f - \beta\mu(X)) = 0$. Since $\mu(Y)$ is monic, $f = \beta\mu(X)$, thus $\underline{f} = 0$. This proves T is faithful.
(ii)For any morphism $f' : TX \to TY$, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & X & \xrightarrow{\mu(X)} & I(X) & \xrightarrow{\pi(X)} & TX & \xrightarrow{0} & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow f' & & \\ 0 & \xrightarrow{0} & Y & \xrightarrow{\mu(Y)} & I(Y) & \xrightarrow{\pi(Y)} & TY & \xrightarrow{0} & 0, \end{array}$$

where the morphism g exists due to that I(X) is $\mathcal{S}-$projective. Now there exists a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & X & \xrightarrow{\mu(X)} & I(X) & \xrightarrow{\pi(X)} & TX & \xrightarrow{0} & 0 \\ & & \downarrow f & & \downarrow I(f) & & \downarrow f'' & & \\ 0 & \xrightarrow{0} & Y & \xrightarrow{\mu(Y)} & I(Y) & \xrightarrow{\pi(Y)} & TY & \xrightarrow{0} & 0 \end{array}$$

where $\underline{f''} = T\underline{f}$. By $(g - I(f))\pi(X) = 0$, we have that there exists a morphism $\alpha : T(X) \to I(Y)$ such that $g - I(f) = \alpha\pi(X)$. Then $(f' - f'')\pi(X) = \pi(Y)(g - I(f)) = \pi(Y)\alpha\pi(X)$. Now we get $(f' - f'' - \pi(Y)\alpha)\pi(X) = 0$. Since $\pi(X)$ is epic, $(f' - f'' - \pi(Y)\alpha) = 0$. Then $\underline{f'} = \underline{f''} = T\underline{f}$. This proves T is full.
(iii)Assume that $Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} TX \xrightarrow{-T\underline{u}} TY$ is a right triangle in $\underline{\mathcal{B}}$. Then we can get the following commutative diagram:

$$\begin{array}{ccccccc} TX & \xrightarrow{-T\underline{u}} & TY & \xrightarrow{-T\underline{v}} & TC_u & \xrightarrow{-T\underline{w}} & T^2X \\ \downarrow 1 & & \downarrow 1 & & \downarrow \underline{l'} & & \downarrow 1 \\ TX & \xrightarrow{-T\underline{u}} & TY & \xrightarrow{-T\underline{g}} & TZ & \xrightarrow{-T\underline{h}} & T^2X \end{array}$$

where the rows are right triangles, and $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} TX$ is a standard right triangle in $\underline{\mathcal{B}}$. By Prop 2.13, $\underline{l'}$ is an isomorphism. Let $\underline{l'}\underline{k'} = 1_{TZ}$ and $\underline{k'}\underline{l'} = 1_{TC_u}$. Since T is full, there exist morphisms $\underline{l} : C_u \to Z$ and $\underline{k} : Z \to C_u$ such that $T\underline{l} = \underline{l'}$, $T\underline{k} = \underline{k'}$. Now we get $T\underline{kl} = T1_{C_u}$ and $T\underline{lk} = T1_Z$. Since T is faithful, l is an isomorphism. Then $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} TX$ is a right triangle in $\underline{\mathcal{B}}$. This proves that $\underline{\mathcal{B}}$ satisfies the Assumption 2.10.

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References

[ABM] I.Assem, A.Beligiannis and N.Marmaridis. Right Triangulated Categories with Right Semi-equivalences. Canandian Mathematical Society Conference Proceeding. Volume**24**, 1998.

[ASS] I.Assem, D.Simson and A.Skowronski. Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory, London Mathematical Society Student Textx, 65. Cambridge University Press, Cambridge, 2006.

[BM] A.Beligiannis and N.Marmaridis. Left triangulated categories arising from contravariantly finite subcategories. Communications in algebra.**22(12)**, 5021-5036, 1994

[BMRRT] A.Buan, R.Marsh, M.Reineke, I.Reiten and G.Todorov. Tilting theory and cluster combinatorics. Advances in Math. **204**, 572-618, 2006.

[BR] A.Beligiannis and I.Reiten, Homological and homotopical aspects of torsion theories, *Memoirs of the AMS* **883** (2007), 426–454.

[CCS] P.Caldero, F.Chapoton and R.Schiffler. Quivers with relations arising from clusters (A_n case). Transactions of the AMS. **358**, 1347-1364, 2006.

[FZ] S.Fomin and A.Zelevinsky. Cluster Algebras I: Foundations. Jour. Amer. Math. Soc. **15**, no. 2, 497–529, 2002.

[H] D.Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. London Mathematical Society, LMN 119, Cambridge, 1988.

[IY] O.Iyama and Y. Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay. modules. Invent. Math. 172 (2008), no.1, 117-168.

[J] P.Jørgensen. Quotients of cluster categories. Proceedings of the Royal Society of Edinburgh.**140A**, 65-81, 2010.

[Ke1] B.Keller. Triangulated orbit categories. Documenta Math. **10**, 551-581, 2005.

[Ke2] B.Keller. Cluster algebras, quiver representations and triangulated categories. In Triangulated categories, volume 375 of LMS Lecture Notes Ser., 375, Cambridge, Pages 76–160. Cambridge Univ. Press, Cambrdge, 2010. See also arXiv: 0807.1960.

[KR] B.Keller and I.Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.

[KZ] S.Koenig and B.Zhu. From triangulated categories to abelian categories– cluster tilting in a general framework. Math. Zeit. 258, 143-160, 2008.

[N] H.Nakaoka Frobenius condition on a pretriangulated category, and triangulation on the associated stable category. Preprint `arXiv:math.cT/1006.1033`

[ZZ] Y.Zhou and B.Zhu. Mutation of torsion pairs in triangulated categories and its geometric realization. arXiv:1105.3521.