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BGP-reflection functors and cluster combinatorics ☆

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Abstract

We define Bernstein–Gelfand–Ponomarev reflection functors in the cluster categories of hereditary algebras. They are triangle equivalences which provide a natural quiver realization of the "truncated simple reflections" on the set of almost positive roots $\Phi_{\geq -1}$ associated with a finite dimensional semi-simple Lie algebra. Combining this with the tilting theory in cluster categories developed in [A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. (in press). math.RT/0402054], we give a unified interpretation via quiver representations for the generalized associahedra associated with the root systems of all Dynkin types (simply laced or non-simply laced). This confirms the Conjecture 9.1 in [A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. (in press). math.RT/0402054] for all Dynkin types.

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1. Introduction

As a model for the combinatorics of a Fomin–Zelevinsky cluster algebra [10,9], the cluster category $\mathcal{C}(H)$ associated with a hereditary algebra H over a field was introduced in [4], see also [5]. It is the orbit category of the (bounded) derived category of H factored by the automorphism $G = [1]\tau^{-1}$, where [1] is the shift functor and τ the Auslander–Reiten translation in the derived category of H. This orbit category is a triangulated category [14]. For when H is the path algebra of a quiver of Dynkin type (simply laced case), it is proved in [4] that there is a one-to-one correspondence between the set of indecomposable objects in $\mathcal{C}(H)$ and the set of cluster variables of the corresponding cluster algebras. This correspondence is given explicitly when the orientation of the quiver is alternating, and under this correspondence, tilting objects correspond to clusters. This was motivated by a previous quiver-theoretic interpretation (using "decorated" quiver representations) for generalized associahedra of simply laced Dynkin type in the sense of Fomin–Zelevinsky [11,6], which was given in [15].

In the combinatorics of cluster algebras, the group of piecewise-linear transformations of the root lattice generated by "truncated simple reflections" σ_i for $i \in I$ (the index set of simple roots) plays an important role as the Weyl

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group in the classical theory of semi-simple Lie algebra. A similar but stronger tool in the representation theory of quivers and hereditary algebras is the so-called Bernstein–Gelfand–Ponomarev reflection functors [2] or APR tilting functors [1]. Since, by [4], the cluster categories provide a successful model for realizing the clusters and associahedra, it is natural to ask whether the BGP-reflection functors can be defined in the cluster categories. These functors defined in the cluster categories should lead to the "truncated simple reflections" on the set of almost positive roots and should be applicable to the clusters and associahedra. One of the motivations for this work comes from [15], where the authors gave a realization of the "truncated simple reflections" in the category of "decorated" quiver representations. Unfortunately their functors are not equivalences.

In this paper, we verify that it is indeed possible to define the BGP-reflection functors in the cluster categories of hereditary algebras (in fact they can be defined in a more general case including the case of root categories (compare [21])). The advantage of our functors (compare with [15]) is that the BGP-reflection functors in cluster categories are triangle equivalences. By applying these equivalences defined in the cluster categories to the set of almost positive roots, we obtain a realization of the "truncated simple reflections" [11]. This enables us to give in a unified way a quiver interpretation for generalized associahedra there. By using this realization, the main ingredients of constructions in Section 3 in [11] follow without much effort from tilting theory developed in [4]. This generalizes the main results on quiver interpretation for generalized associahedra of the simply laced case in [15] and confirms the Conjecture 9.1. [4] in all Dynkin types.

2. BGP-reflection functors in orbit triangulated categories

It is well known that the orbit category $D^b(H)/G$ of the derived category of a finite dimensional hereditary algebra H is a triangulated category in which the images of triangles in $D^b(H)$ under the natural projection are still triangles when G is an automorphism satisfying some specific conditions (the conditions $(g1)$, $(g2)$ below) [14]. When $G = [1]\tau^{-1}$, the orbit category $D^b(H)/G$ is called the cluster category of H. We recall some basics on orbit triangulated categories from [14] and basics on the cluster categories from [4,3].

Let $\mathcal{H}$ be a hereditary category with Serre duality and with finite dimensional Hom-spaces and Ext-spaces over a field K. Denote by $\mathcal{D} = D^b(\mathcal{H})$ the bounded derived category of $\mathcal{H}$ with shift functor [1]. For any category $\mathcal{E}$, we will denote by ind $\mathcal{E}$ the subcategory of isomorphism classes of indecomposable objects in $\mathcal{E}$; depending on the context we shall also use the same notation to denote the set of isomorphism classes of indecomposable objects in $\mathcal{E}$. For any T in $\mathcal{H}$, we denote the subcategory of $\mathcal{H}$ consisting of direct summands of direct sums of finitely many copies of T by add T. Note that add H denotes the category of projective H-modules.

Let $G: \mathcal{D} \to \mathcal{D}$ be a standard equivalence, i.e. G is isomorphic to the derived tensor product

$$- \otimes_A X : D^b(A) \to D^b(A)$$

for some complex X of A—A-bimodules. We also assume that G satisfies the following properties:

(g1) For each U in ind $\mathcal{H}$, only a finite number of objects $G^n U$, where $n \in \mathbf{Z}$, lie in ind $\mathcal{H}$.
(g2) There is some $N \in \mathbf{N}$ such that $\{U[n] \mid U \in \text{ind } \mathcal{H}, n \in [-N, N]\}$ contains a system of representatives of the orbits of G on ind $\mathcal{D}$.

We denote by $\mathcal{D}/G$ the corresponding factor category. The objects are by definition the G-orbits of objects in $\mathcal{D}$, and the morphisms are given by

$$\text{Hom}_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) = \oplus_{i \in \mathbf{Z}} \text{Hom}_{\mathcal{D}}(G^i X, Y).$$

Here X and Y are objects in $\mathcal{D}$, and $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding objects in $\mathcal{D}/G$ (although we shall sometimes write such objects simply as X and Y). The orbit category $\mathcal{D}/G$ is a Krull–Schmidt category [4] and also a triangulated category [14]. The natural functor $\pi: \mathcal{D} \to \mathcal{D}/G$ is a covering functor of triangulated categories in the sense that π is a covering functor and a triangle functor [20]. The shift in $\mathcal{D}/G$ is induced by the shift in $\mathcal{D}$, and is also denoted by [1]. In both cases we write as usual $\text{Hom}(U, V[1]) = \text{Ext}^1(U, V)$. We then have

$$\text{Ext}^1_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) = \oplus_{i \in \mathbf{Z}} \text{Ext}^1_{\mathcal{D}}(G^i X, Y),$$

where X, Y are objects in $\mathcal{D}$ and $\widetilde{X}, \widetilde{Y}$ are the corresponding objects in $\mathcal{D}/G$.

We shall mainly consider the special choice of functor $G = [1]\tau^{-1}$, where τ is the Auslander–Reiten translation in $\mathcal{D}$ and $\mathcal{H} = H$-mod is the category of finite dimensional left modules over finite dimensional hereditary algebras H. In this case the factor category $\mathcal{D}/G$ is called the cluster category of H, which is denoted by $\mathcal{C}(H)$. It is not difficult to see that ind $\mathcal{C}(H) = \{\widetilde{X} \mid X \in \text{ind } (H - \text{mod} \vee H[1])\}$ [4].

Now we recall the representations of a species of a valued graph from [8]. A valued graph $(\Gamma, \mathbf{d})$ is a finite set Γ (of vertices) together with non-negative integers d_{ij} for all pairs $i, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i\in\Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i, \quad \text{for all } i,\ j \in \Gamma.$$

A pair $\{i, j\}$ of vertices is called an edge of $(\Gamma, \mathbf{d})$ if $d_{ij} \neq 0$. An orientation Ω of a valued graph $(\Gamma, \mathbf{d})$ is given by prescribing for each edge $\{i, j\}$ of $(\Gamma, \mathbf{d})$ an order (indicated by an arrow $i \to j$). Given an orientation Ω and a vertex $k \in \Gamma$, we can define a new orientation $s_k\Omega$ of $(\Gamma, \mathbf{d})$ by reversing the direction of arrows along all edges containing k. A vertex $k \in \Gamma$ is called a sink (resp., source) with respect to Ω if there are no arrows starting (resp., ending) at vertex k.

Let K be a field and $(\Gamma, \mathbf{d}, \Omega)$ a valued quiver. From now on, we shall always assume that Γ contains no cycles. Let $\mathbf{M} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a reduced K-species of type $(\Gamma, \mathbf{d}, \Omega)$; that is, for all $i, j \in \Gamma$, ${}_iM_j$ is an $F_i - F_j$-bimodule, where F_i and F_j are division rings which are finite dimensional vector spaces over K and $\dim({}_iM_j)_{F_j} = d_{ij}$ and $\dim_K F_i = \varepsilon_i$. A K-representation $V = (V_i, \varphi_\alpha)$ of $(\mathbf{M}, \Gamma, \Omega)$ consists of an F_i-vector space $V_i, i \in \Gamma$, and of an F_j-linear map ${}_j\varphi_i : V_i \otimes {}_iM_j \to V_j$ for each arrow $i \to j$. Such a representation is called finite dimensional if $\sum_{i\in\Gamma} \dim_K V_i < \infty$. The category of finite dimensional representations of $(\mathbf{M}, \Gamma, \Omega)$ over K is denoted by rep$(\mathbf{M}, \Gamma, \Omega)$.

Now we fix a K-species $\mathbf{M}$ of type $(\Gamma, \mathbf{d}, \Omega)$. In the rest of the paper, we always speak of the valued quiver $(\mathbf{M}, \Gamma, \Omega)$ instead of $(\Gamma, \mathbf{d}, \Omega)$. Given a sink, or a source k of the valued quiver $(\mathbf{M}, \Gamma, \Omega)$, we are going to recall the Bernstein–Gelfand–Ponomarev reflection functor (shortened to the BGP-reflection functor) $S_k^{\pm}$:

$$S_k^+ : \text{rep}(\mathbf{M}, \Gamma, \Omega) \longrightarrow \text{rep}(\mathbf{M}, \Gamma, s_k\Omega),$$

and respectively

$$S_k^- : \text{rep}(\mathbf{M}, \Gamma, \Omega) \longrightarrow \text{rep}(\mathbf{M}, \Gamma, s_k\Omega).$$

For any representation $V = (V_i, \phi_\alpha)$ of $(\mathbf{M}, \Gamma, \Omega)$, the image of it under S_k^+ is, by definition, $S_k^+V = (W_i, \psi_\alpha)$, a representation of $(\mathbf{M}, \Gamma, s_k\Omega)$, where $W_i = V_i$ when $i \neq k$; and W_k is the kernel in the diagram:

$$0 \longrightarrow W_k \xrightarrow{({}_j\chi_k)_j} \oplus_{j\in\Gamma} V_j \otimes {}_jM_k \xrightarrow{({}_k\phi_j)_j} V_k \qquad (*)$$

$\psi_\alpha = \phi_\alpha$ when the ending vertex of α is not k; and when the ending vertex of α is k, $\psi_{s_k\alpha} = {}_j\bar{\chi}_k : W_k \otimes {}_kM_j \to X_j$, where ${}_j\bar{\chi}_k$ corresponds to ${}_j\chi_k$ under the isomorphism $\text{Hom}_{F_j}(W_k \otimes {}_kM_j, V_j) \approx \text{Hom}_{F_i}(W_k, V_j \otimes {}_jM_i)$.

If $\mathbf{f} = (f_i) : V \to V'$ is a morphism in rep$(\mathbf{M}, \Gamma, \Omega)$, then $S_k^+(\mathbf{f}) = \mathbf{g} = (g_i)$, where $g_i = f_i$ for $i \neq k$ and $g_k : W_k \to W_k'$ as the restriction of $\oplus_{j\in\Gamma}(f_j \otimes 1)$ given in the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & W_k & \xrightarrow{({}_j\chi_k)_j} & \oplus_{j\in\Gamma} V_j \otimes {}_jM_k & \xrightarrow{({}_k\phi_j)_j} & V_k \\
\downarrow & & \downarrow g_k & & \downarrow \oplus_j(f_j\otimes 1) & & \downarrow f_k \\
0 & \longrightarrow & W_k' & \xrightarrow{({}_j\chi_k')_j} & \oplus_{j\in\Gamma} V_j' \otimes {}_jM_k & \xrightarrow{({}_k\phi_j')_j} & V_k'
\end{array}$$

If k is a source, the definition of $S_k^- V$ is dual to that of $S_k^+ V$; we omit the details and refer the reader to [8].

Let k be a sink and P_i the indecomposable projective representation of $(\mathbf{M}, \Gamma, \Omega)$ corresponding to vertex $i \in \Gamma$. Let $T = \oplus_{i\in\Gamma-k} P_i \oplus \tau^{-1}P_k$ and $H = \oplus_{i\in\Gamma} P_i$. Then T is a tilting module in rep$(\mathbf{M}, \Gamma, \Omega)$ [1], $S_k^+ = \text{Hom}(T, -)$. It induces an equivalence from add T to add H' where H' is the tensor algebra of $(\mathbf{M}, \Gamma, s_k\Omega)$, and induces a triangle equivalence $\text{Hom}_H(T, -) : K^b(\text{add } T) \to K^b(\text{add } H')$. As in [12], the composition of functors indicated as the following arrows:

$$\mathbf{K}^b(\text{add } T) \hookrightarrow \mathbf{K}^b(H) \to D^b(H)$$

is a triangle equivalence. It is easy to see that S_k^+ and S_k^- commutes with the shift functor [1]. Since $D^b(H)$ has Auslander–Reiten triangles and S_k^+ or S_k^- sends AR-triangles to AR-triangles (compare to Theorem 4.6 in Chapter I in [12]), S_k^+ and S_k^- commute with τ.

We summarize these facts in the following lemma.

Lemma 2.1. *Let k be a sink (resp., source) of a valued quiver $(\mathbf{M}, \Gamma, \Omega)$. Then S_k^+ (resp., S_k^-) induces a triangle equivalence from $D^b(H)$ to $D^b(H')$ which is denoted also by S_k^+ (resp., S_k^-); and $S_k^{\pm}$ commutes with the shift functor* [1] *and the AR-translation τ.*

In the following, we assume that the standard equivalence $G : D^b(H) \to D^b(H)$ satisfies the conditions $(g1)$ and $(g2)$. Then $G' = S_k^+ G S_k^-$ is also a standard equivalence of $D^b(H')$ which satisfies $(g1)$ and $(g2)$. We define a functor $R(S_k^+)$ from $D^b(H)/G$ to $D^b(H')/G'$ as follows: Let $\widetilde{X} \in D^b(H)/G$ with $X \in D^b(H)$. Let X_T be one of the complexes in $C^b(\text{addT})$ which are quasi-isomorphic to X, where $C^b(\text{addT})$ denotes the category of complexes with finitely many non-zero components and all components belong to add T. We set $R(S_k^+)(\widetilde{X}) = \widetilde{S_k^+(X_T)}$. For morphism $\tilde{f} : \widetilde{X} \to \widetilde{Y}$, we set $R(S_k^+)(\tilde{f}) : \widetilde{S_k^+(X_T)} \to \widetilde{S_k^+(Y_T)}$ to be the map $\widetilde{S_i^+(f_T)}$, where f_T is the one induced from f under the quasi-isomorphism from X to X_T.

We prove that $R(S_k^+)$ is a triangle equivalence (compare Section 9.4 in [14]).

Theorem 2.2. *Let k be a sink (resp., source) of a valued quiver $(\mathbf{M}, \Gamma, \Omega)$. Then $R(S_k^+)$ (resp., $R(S_k^-)$) is a triangle equivalence from $D^b(H)/G$ to $D^b(H')/G'$.*

Proof. First of all, we verify that the definition is well defined: For $\widetilde{X} = \widetilde{Y} \in D^b(H)/G$ with $X, Y \in D^b(H)$, we have that $Y = G^i(X)$ for some integer i. It follows that $Y_T = G^i(X_T)$ in $D^b(H)$. By applying $\widetilde{S_k^+}$ to the two complexes above, we have that $S_k^+(Y_T) = S_k^+ G^i S_k^-(S_k^+(X_T)) = G'^i(S_k^+(X_T))$. It follows that $\widetilde{S_k^+(Y_T)} = \widetilde{S_k^+(X_T)}$, i.e. $R(S_k^+)(\widetilde{X}) = R(S_k^+)(\widetilde{Y})$. The action of $R(S_k^+)$ on morphisms is induced by S_k^+ on morphisms in $D^b(H)$ in the way indicated in the following commutative diagram:

$$\begin{array}{ccc}
\oplus_{i\in\mathbf{Z}}\mathrm{Hom}_{D^b(H)}(G^i(X_{\mathrm{T}}), Y_{\mathrm{T}}) & \xrightarrow{S_k^+} & \oplus_{i\in\mathbf{Z}}\mathrm{Hom}_{D^b(H')}(G'^i(S_k^+(X_{\mathrm{T}})), S_k^+(Y_{\mathrm{T}})) \\
\wr\downarrow & & \downarrow\wr \\
\mathrm{Hom}_{D^b(H)/G}(\widetilde{X}, \widetilde{Y}) & \xrightarrow{R(S_k^+)} & \mathrm{Hom}_{D^b(H')/G'}(R(S_k^+)(\widetilde{X}), R(S_k^+)(\widetilde{Y}))
\end{array}$$

It is easy to verify that $R(S_k^+)$ and $R(S_k^-)$ satisfy: $R(S_k^+) \circ R(S_k^-) \approx \mathrm{id}_{D^b(H')/G'}$ and $R(S_k^-) \circ R(S_k^+) \approx \mathrm{id}_{D^b(H)/G}$. These show that $R(S_k^+)$ and $R(S_k^-)$ are equivalences. Now by using the result in Section 9.4 of [14], we have that $R(S_k^+)$ sends triangles in $D^b(H)/G$ to triangles in $D^b(H')/G'$. Therefore $R(S_k^+)$ is a triangle equivalence. The proof is finished. □

When $G = \tau^{-1}[1]$, we have the triangle equivalence $R(S_k^+)$ from the cluster category $\mathcal{C}(\Omega) = D^b(H)/G$ to $\mathcal{C}(s_k\Omega) = D^b(H')/G$. And when $G = [2]$, we have the triangle equivalence from the root category $D^b(H)/[2]$ to the root category $D^b(H')/[2]$ (compare [21]).

Let P_i (resp., P_i') be the indecomposable projective representations in H-mod (resp., H'-mod) corresponding to the vertex $i \in \Gamma_0$, E_j (resp., E_j') be the simple H-module (resp., simple H'-module) corresponding to the vertex j.

Corollary 2.3. *Let k be a sink of a valued quiver $(\mathbf{M}, \Gamma, \Omega)$. Then $R(S_k^+)$ is a triangle equivalence from $\mathcal{C}(\Omega)$ to $\mathcal{C}(s_k\Omega)$. Moreover for $X \in$ ind H, $R(S_k^+)(\widetilde{X}) = \begin{cases} \widetilde{P_k'[1]} & \text{if } X \cong E_k \\ \widetilde{S_k^+(X)} & \text{otherwise;} \end{cases}$ and for $j \neq k$, $R(S_k^+)(\widetilde{P_j}[1]) = \widetilde{P_j'}[1]$, and $R(S_k^+)(\widetilde{P_k}[1]) = \widetilde{E_k'}$.*

Proof. From Theorem 2.2., $R(S_k^+)$ is a triangle equivalence from the cluster category $\mathcal{C}(\Omega)$ to $\mathcal{C}'(s_k\Omega)$. Now we prove that $R(S_k^+)(\widetilde{E_k}) = \widetilde{P'_k}[1]$. Since k is a sink, we have AR-sequence (*): $0 \to E_k \to X \to \tau^{-1}E_k \to 0$ in H-mod with X and $\tau^{-1}E_k$ being in add T [1] or [18]. Since S_k^+ is a left exact functor, we have the exact sequence $0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k)$ in H'-mod, in which the cokernel of the injective map is E'_k. As the stalk complex of degree 0, $E_k^\bullet$ is isomorphic to the complex: $\cdots \to 0 \to X \to \tau^{-1}E_k \to 0 \to \cdots$ in $D^b(H)$. By applying S_k^+ to the complex above, we have that $S_k^+(E_k^\bullet) = \cdots \to 0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k) \to 0 \to \cdots$. It follows that the complex $\cdots \to 0 \to S_k^+(X) \to S_k^+(\tau^{-1}E_k) \to 0 \to \cdots$ is quasi-isomorphic to the stalk complex $\widetilde{E_k'^\bullet[-1]}$ of degree -1. It follows that $R(S_k^+)(\widetilde{E_k}) = \widetilde{E'_k}[-1]$. Since $\tau\widetilde{P'_k} = \widetilde{E'_k}[-1]$, $R(S_k^+)(\widetilde{E_k}) = \tau\widetilde{P'_k} = (\tau\widetilde{[-1]})(P'_k)[1] = \widetilde{P'_k}[1]$. In the derived category $D^b(H)$, we have that $S_k^+(P_i) = P'_i$ for any $i \neq k$, $S_k^+(E_k[1]) = E'_k$. It follows that $R(S_k^+)(\widetilde{P_i}) = \widetilde{P'_i}$ for any $i \neq k$ and $R(S_k^+)(\widetilde{P_k}[1]) = \widetilde{E'_k}$. The proof is finished. □

Remark 2.4. We leave the dual statement for a source k to the reader.

Definition 2.5. When k is a sink (resp., source) of $(\mathbf{M}, \Gamma, \Omega)$, the functor $R(S_k^+)$ (resp., $R(S_k^-)$) of Corollary 2.3 is called a BGP-reflection functor in the cluster category $\mathcal{C}(\Omega)$.

Let Γ be a classical Dynkin quiver, i.e. one of the types ADE. Then the automorphisms of $D^b(H)$ are of the form $[n]$, τ^n, or of the form $[n]\tau^m$ for any $m, n \in \mathbf{Z}$ (compare [20]).

Corollary 2.6. *Let Γ be a Dynkin quiver and k a sink (resp., source) of it. Then for any automorphism G of $D^b(H)$ which is not of the form $([1]\tau^m)^t$, where t is an integer and $m = (n+1)/2$ if the underlying diagram of Γ is of type A_n; or $m = 6$ if the underlying diagram of Γ is of type E_6, $R(S_k^+)$ (resp., $R(S_k^-)$) can be defined and it is a triangle equivalence from $D^b(H)/G$ to $D^b(H')/G'$.*

Proof. It follows from Proposition 3.3.2 in [20] that any automorphism G of $D^b(H)$ is generated by τ and [1]. Therefore G commutes with S_k^+. For automorphism G indicated in the corollary, G satisfies the conditions $(g1)$ and $(g2)$; hence the orbit category $D^b(H)/G$ exists [14]. Then by Theorem 2.2, $R(S_k^+)$ exists and is a triangle equivalence. The proof is finished. □

3. Applications to cluster combinatorics

In this section, we always assume that H is the tensor algebra of a valued quiver $(\mathbf{M}, \Gamma, \Omega)$ over a field K, with underlying graph Γ, where Γ is not necessarily connected. We denote by $\mathcal{A} = \mathcal{A}(\Gamma)$ the corresponding cluster algebra when Γ is of Dynkin type (simply laced or non-simply laced), by Φ the set of roots of the corresponding Lie algebra, and by $\Phi_{\geq -1}$ the set of almost positive roots, i.e. the positive roots together with the negatives of the simple roots. The elements of $\Phi_{\geq -1}$ are in 1–1 correspondence with cluster variables of $\mathcal{A}$ (Theorem 1.9. [10]); such 1–1 correspondence is denoted by $\mathcal{P}$. Fomin and Zelevinsky [11] associate a non-negative integer $(\alpha \parallel \beta)$, known as the compatibility degree, with each pair α, β of almost positive roots. This is defined in the following way. Let s_i be the Coxeter generator of the Weyl group of Φ corresponding to i, and let σ_i be the permutation of $\Phi_{\geq -1}$ defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j,\ j \neq i \\ s_i(\alpha) & \text{otherwise.} \end{cases} \tag{3.1}$$

The σ_i's are called "truncated simple reflections" of $\Phi_{\geq -1}$. They are among the main ingredients of constructions in [11] (see also [15]). Let $\Gamma = \Gamma^+ \sqcup \Gamma^-$ be a partition of the set of vertices of Γ into completely disconnected subsets and define

$$\tau_\pm = \prod_{i \in \Gamma^\pm} \sigma_i. \tag{3.2}$$

Denote by $[\beta : \alpha_i]$ the coefficient of α_i in the expression of β in simple roots $\alpha_1, \ldots, \alpha_n$. Then $(\parallel)$ is uniquely defined by the following two properties:

$$(-\alpha_i \parallel \beta) = \max([\beta : \alpha_i], 0), \tag{3.3}$$

$$(\tau_{\pm}\alpha \parallel \tau_{\pm}\beta) = (\alpha \parallel \beta), \tag{3.4}$$

for any $\alpha, \beta \in \Phi_{\geq -1}$, any $i \in \Gamma$.

A pair α β in $\Phi_{\geq -1}$ are called compatible if $(\alpha \parallel \beta) = 0$. Associated with the finite root system Φ, the simplicial complex $\Delta(\Phi)$ is defined in [11]. $\Delta(\Phi)$ has $\Phi_{\geq -1}$ as the set of vertices; its simplices are mutually compatible subsets of $\Phi_{\geq -1}$. The maximal simplices of $\Delta(\Phi)$ are called the clusters associated with Φ. This simplicial complex $\Delta(\Phi)$ is called the generalized associahedron (compare [5,6,10,11]).

In this section, we will first show that the truncated simple reflections σ_i on $\Phi_{\geq -1}$ can be realized by the BGP-reflection functors $R(S_i^+)$ in the corresponding cluster category. Then, by using these BGP-reflection functors and combining tilting theory in cluster categories developed in [4], we give a unified quiver interpretation of certain combinatorics about clusters associated with arbitrary root systems of (simply laced or non-simply laced) semi-simple Lie algebras in [11]. This extends, in a different way, the quiver-theoretic interpretation of certain combinatorics about clusters in the simply laced case given by Marsh et al. in [15]. They use decorated representations.

Let $\{e_i \mid i \in \Gamma\}$ be a complete set of primitive idempotents of a hereditary algebra H. For any subgraph J of Γ, we set $I = HeH$, the hereditary ideal of H, where $e = \sum_{i \in \Gamma - J} e_i$. Then the quotient algebra $A = H/I$ has a complete set of primitive idempotents $\bar{e}_i i \in J$. A-mod is a full subcategory of H-mod consisting of H-modules annihilated by I or, in other words, consisting of H-modules whose composition factors are E_i with $i \in J$. It follows from [8,16] that $\mathrm{Ext}^i_A(X, Y) = \mathrm{Ext}^i_H(X, Y)$ for any $X, Y \in A$-mod and any i. It follows that A is also a hereditary algebra which is Morita equivalent to the tensor algebra of $(\mathbf{M}|_J, J, \Omega|_J)$. These facts are summarized in the following proposition.

Proposition 3.1. *$D^b(A)$ is a triangulated subcategory of $D^b(H)$ and $\mathcal{C}(A)$ is a triangulated subcategory of $\mathcal{C}(H)$.*

Proof. A is hereditary and $\mathrm{Ext}^i_A(X, Y) = \mathrm{Ext}^i_H(X, Y)$ for any $X, Y \in \mathrm{mod}A$ and any i. This gives us that $D^b(A) \subseteq D^b(H)$ is a full triangulated subcategory of $D^b(H)$. It follows that the cluster category $\mathcal{C}(A)$ is a full triangulated subcategory of $\mathcal{C}(H)$. The proof is finished. $\square$

We recall the notation of exceptional sets and of tilting sets in $\mathcal{C}(\Omega)$ in [4]. A subset B of ind $\mathcal{C}(\Omega)$ is called exceptional if $\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(X, Y) = 0$ for any $X, Y \in B$. A maximal exceptional set is called a tilting set. A subset of $\mathcal{C}(\Omega)$ is a tilting set if and only the direct sum of all objects in B is a basic tilting object [4]. Then any tilting set contains exactly $|\Gamma|$ many objects. One can associate with $\mathcal{C}(\Omega)$ a simplicial complex $\Delta(\Omega)$ as follows: $\Delta(\Omega)$ has ind $\mathcal{C}(\Omega)$ as the set of vertices; its simplices are the exceptional sets in ind $\mathcal{C}(\Omega)$. It is easy to see that its maximal simplices are exactly tilting sets [4]. One can also associate with $\mathcal{C}(\Omega)$ a tilting graph Δ_Ω whose vertices are the basic tilting objects, and where there is an edge between two vertices if the corresponding tilting objects have all but one indecomposable summands in common. Tilting graphs associated with a hereditary algebra were studied by Riedtmann and Schofield [17], Unger [19], and also Happel and Unger [13].

In general, BGP-reflection functors preserve exceptional sets and tilting sets.

Proposition 3.2. *Let k be a sink (resp., source) of a valued quiver $(\mathbf{M}, \Gamma, \Omega)$ of any type. Then the BGP-reflection functor $R(S_k^+)$ (resp., $R(S_k^-)$) gives a 1-1 correspondence from the set of exceptional sets in* ind $\mathcal{C}(\Omega)$ *to that in* ind $\mathcal{C}(s_k\Omega)$*; under this correspondence, tilting sets go to tilting sets. In particular if $(\mathbf{M}, \Gamma, \Omega)$ and $(\mathbf{M}, \Gamma, \Omega')$ are two valued quivers of the same type Γ, then the simplicial complexes $\Delta(\Omega)$ and $\Delta(\Omega')$ are isomorphic and the tilting graphs Δ_Ω and $\Delta_{\Omega'}$ are isomorphic.*

Proof. Suppose k is a sink. Since $R(S_k^+)$ and $R(S_k^-)$ are inverse equivalences between $\mathcal{C}(\Omega)$ and $\mathcal{C}(s_k\Omega)$,

$$\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(X, Y) = \mathrm{Ext}^1_{\mathcal{C}(s_k\Omega)}(R(S_k^+)(X), R(S_k^+)(Y)), \quad \text{for any } X, Y \in \mathcal{C}(\Omega).$$

It follows that $R(S_k^+)$ and $R(S_k^-)$ give inverse maps between the sets of exceptional sets in ind $\mathcal{C}(\Omega)$ and in ind $\mathcal{C}(s_k\Omega)$. An exceptional set is a tilting set if and only if so is its image under $R(S_k^+)$. For any two valued quivers with the same graph, one can get an admissible sequence $i_1, \ldots, i_n$ such that $\Omega' = s_{i_n} \cdots s_{i_1}\Omega$ with i_k is the sink of $s_{i_{k-1}} \cdots s_{i_1}\Omega$. For each k, we have that the fact of equivalence of $R(S_{i_k}^+)$ implies $\Delta_{s_{i_{k-1}} \cdots s_{i_1}\Omega} \simeq \Delta_{s_{i_k}s_{i_{k-1}} \cdots s_{i_1}\Omega}$ and $\Delta(s_{i_{k-1}} \cdots s_{i_1}\Omega) \simeq \Delta(s_{i_k}s_{i_{k-1}} \cdots s_{i_1}\Omega)$. Therefore $\Delta_\Omega \simeq \Delta_{\Omega'}$ and $\Delta(\Omega) \simeq \Delta(\Omega')$. The proof is finished. $\square$

Now we recall the decorated quiver representations from [15]. Let Q be a Dynkin quiver with vertices Q_0 and arrows Q_1. The "decorated" quiver $\widetilde{Q}$ is the quiver Q with an extra copy $Q_0^- = \{i_- : i \in Q_0\}$ of the vertices of

Q (with no arrows incident with the new copy). A module M over $k\widetilde{Q}$ can be written in the form $M^+ \oplus V$, where $M^+ = \oplus_{i\in Q_0} M_i^+$ is a KQ-module, and $V = \oplus_{i\in Q_0} V_i$ is a Q_0-graded vector space over K. Its signed dimension vector, $\mathbf{sdim}(M)$, is the element of the root lattice of the Lie algebra of type Q given by

$$\mathbf{sdim}(M) = \sum_{i\in Q_0} \dim(M_i^+)\alpha_i - \sum_{i\in Q_0} \dim(V_i)\alpha_i,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the simple roots. By Gabriel's Theorem, the indecomposable objects of $K\widetilde{Q}$-mod are parameterized, via $\mathbf{sdim}$, by the almost positive roots, $\Phi_{\geq -1}$, of the corresponding Lie algebra. The positive roots correspond to the indecomposable KQ-modules, and the negative simple roots correspond to the simple modules associated with the new vertices. We denote the simple module corresponding to the vertex i_- by E_i^-. Let $M = M^+ \oplus V$ and $N = N^+ \oplus W$ be two $K\widetilde{Q}$-modules. The symmetrized Ext^1-group for this pair of modules is defined to be

$$E_{KQ}(M, N) := \mathrm{Ext}^1_{KQ}(M^+, N^+) \oplus \mathrm{Ext}^1_{KQ}(N^+, M^+) \oplus \mathrm{Hom}^{Q_0}(M^+, W) \oplus \mathrm{Hom}^{Q_0}(V, N^+),$$

where Hom^{Q_0} denotes homomorphisms of Q_0-graded vector spaces.

The map ψ_Q from ind $\mathcal{C}(KQ)$ to the set of isomorphism classes of indecomposable $K\widetilde{Q}$-modules is defined in [4] as follows. Let $\widetilde{X} \in$ ind $\mathcal{C}(KQ)$. It can be assumed that one of the following cases holds:

1. X is an indecomposable KQ-module M^+.
2. $X = P_i[1]$ where P_i is the indecomposable projective KQ-module corresponding to vertex $i \in Q_0$.

We define $\psi_Q(\widetilde{X})$ to be M^+ in Case (1), and to be E_i^- in Case (2).

Then the map ψ_Q is a bijection between ind $\mathcal{C}(KQ)$ and the set of isomorphism classes of indecomposable $K\widetilde{Q}$-modules (i.e. indecomposable decorated representations). If we make the definition $\gamma_Q := \mathbf{sdim} \circ \psi_Q$, then it is a bijection between ind $\mathcal{C}(KQ)$ and $\Phi_{\geq -1}$ (and thus induces a bijection between ind $\mathcal{C}(KQ)$ and the set of cluster variables). For $\alpha \in \Phi_{\geq -1}$ we denote by $M_Q(\alpha)$ the element of ind $\mathcal{C}(KQ)$ such that $\gamma_Q(M_Q(\alpha)) = \alpha$. It was proved in [4] that

$$E_{KQ}(\psi_Q(\widetilde{X}), \psi_Q(\widetilde{Y})) \simeq \mathrm{Ext}^1_{\mathcal{C}(KQ)}(\widetilde{X}, \widetilde{Y}), \quad \text{for } X, Y \in \mathcal{D}.$$

Now we return to the general case. Let $(\mathbf{M}, \Gamma, \Omega)$ be a Dynkin valued quiver. We extend first the bijection γ_Q to the general case $\gamma_{(\mathbf{M},\Gamma,\Omega)}$ (which is denoted for simplicity by γ_Ω) from ind $\mathcal{C}$ to $\Phi_{\geq -1}$ by making the following definition: Let $X \in$ ind $(H - \mathrm{mod} \vee H[1])$.

$$\gamma_\Omega(\widetilde{X}) = \begin{cases} \mathbf{dim}\, X & \text{if } X \in \text{ind } H; \\ -\mathbf{dim}\, E_i & \text{if } X = P_i[1], \end{cases}$$

where $\mathbf{dim}\, X$ denotes the dimension vector of H-module X. It is easy to see that the map γ_Ω is a bijection and it is dependent on the orientation Ω of Γ.

Let $\alpha, \beta \in \Phi_{\geq -1}$ and $M(\alpha)$, $M(\beta)$ the indecomposable objects in $\mathcal{C}(\Omega)$ corresponding to α, β under the bijection γ_Ω. For any pair of objects $M,\ N$ in $\mathcal{C}(\Omega)$, $\mathrm{Hom}_{\mathcal{C}(\Omega)}(M, N)$ is a left $\mathrm{End}_{\mathcal{C}(\Omega)}M$-module (here the composition of maps $f : X \to Y$ and $g : Y \to Z$ is $f \circ g : X \to Z$). Therefore $\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M, N)$ is a left $\mathrm{End}_{\mathcal{C}(\Omega)}M$-module. For an algebra A and an A-module X, we denote by $l({}_A X)$ the length of the A-module X. It is easy to see that $R(S_k^+)$ induces an isomorphism from $\mathrm{End}_{\mathcal{C}(\Omega)}M$ to $\mathrm{End}_{\mathcal{C}(s_k\Omega)}(R(S_k^+)M)$. Under this isomorphism, $R(S_k^+)$ induces an $\mathrm{End}_{\mathcal{C}(\Omega)}M$ ($\cong \mathrm{End}_{\mathcal{C}(s_k\Omega)}(R(S_k^+)M)$)-module isomorphism between $\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M, N)$ and $\mathrm{Ext}^1_{\mathcal{C}(s_k\Omega)}(R(S_k^+)(M), R(S_k^+)(N))$, for any sink k. Similar isomorphisms hold if k is a source.

Definition 3.3. For any two almost positive roots $\alpha, \beta \in \Phi_{\geq -1}$, we define the Ω-compatibility degree $(\alpha \parallel \beta)_\Omega$ of α, β by

$$(\alpha \parallel \beta)_\Omega = l\left({}_{\mathrm{End}M(\alpha)}\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M(\alpha), M(\beta))\right).$$

Note that if (Γ, Ω) is a simply laced Dynkin quiver, then the Ω-compatibility degree $(\alpha \parallel \beta)_\Omega$ of α, β equals $\dim_K \mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M(\alpha), M(\beta))$.

We now prove the first main result of the paper.

Theorem 3.4. *Let* $(\mathbf{M}, \Gamma, \Omega)$ *be a valued Dynkin quiver and* k *a sink (resp., source). Then we have the commutative diagram:*

$$\begin{array}{ccc} \Phi_{\geq -1} & \xrightarrow{\sigma_k} & \Phi_{\geq -1} \\ \gamma_\Omega^{-1}\big\downarrow & & \big\downarrow\gamma_{s_k\Omega}^{-1} \\ \operatorname{ind}\mathcal{C}(\Omega) & \xrightarrow[\text{(resp., } R(S_k^-))]{R(S_k^+)} & \operatorname{ind}\mathcal{C}(s_k\Omega) \end{array}$$

Moreover $(\alpha \parallel \beta)_\Omega = (\sigma_k(\alpha) \parallel \sigma_k(\beta))_{s_k\Omega}$.

Proof. Let $\alpha \in \Phi_{\geq -1}$ be a positive root. Then $\sigma_k(\alpha) = -\alpha_k$ when $\alpha = \alpha_k$, and $\sigma_k(\alpha) = s_k(\alpha)$ when α is a positive root other than α_k. It follows that $\gamma_{s_k\Omega}^{-1}\sigma_k(\alpha)$ is $\widetilde{P_k'}[1]$ or $\widetilde{S_k^+(X)}$, respectively, where X is the unique indecomposable representation with **dim** $X = \alpha$ which does exist by Gabriel's theorem [7,18]. On the other hand, $R(S_k^+)\gamma_\Omega^{-1}(\alpha)$ equals $R(S_k^+)(\widetilde{E_k})$ or $R(S_k^+)(\widetilde{X})$ according as α is a simple root α_k or not. Then it follows from Corollary 2.3 that $\gamma_{s_k\Omega}^{-1}\sigma_k(\alpha) = R(S_k^+)\gamma_\Omega^{-1}(\alpha)$. We now prove the equality above for α a negative root. Let $\alpha = -\alpha_i$ for $i \in \Gamma$. Then we have that

$$\gamma_{s_k\Omega}^{-1}\sigma_k(-\alpha_i) = \begin{cases} \widetilde{E_k'} & \text{if } i = k \\ \widetilde{P_i'[1]} & \text{if } i \neq k. \end{cases}$$

Again from Corollary 2.4, we have that $R(S_k^+)\gamma_\Omega^{-1}(-\alpha_k) = R(S_k^+)(\widetilde{P_k[1]}) = \widetilde{E_k'[1]}$ and for $i \neq k$, $R(S_k^+)\gamma_\Omega^{-1}(-\alpha_i) = R(S_k^+)(\widetilde{P_i[1]}) = \widetilde{P_i'[1]}$. This finishes the proof of the commutativity of the diagram. By definition, $(\sigma_k(\alpha) \parallel \sigma_k(\beta))_{s_k\Omega} = l({}_{\mathrm{End}M(\sigma_k(\alpha))}\mathrm{Ext}^1_{\mathcal{C}(s_k\Omega)}(M(\sigma_k(\alpha)), M(\sigma_k(\beta))))$. On the other hand, it follows from the commutative diagram which is proved above that

$$(\sigma_k(\alpha) \parallel \sigma_k(\beta))_{s_k\Omega} = l\left({}_{\mathrm{End}R(S_k^+)(M(\alpha))}\mathrm{Ext}^1_{\mathcal{C}(s_k\Omega)}(R(S_k^+)(M(\alpha)), R(S_k^+)(M(\beta)))\right).$$

The right hand side of the equality equals $l({}_{\mathrm{End}M(\alpha)}\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M(\alpha), M(\beta)))$ since $R(S_k^+)$ is a triangle equivalence. Therefore $(\alpha \parallel \beta)_\Omega = (\sigma_k(\alpha) \parallel \sigma_k(\beta))_{s_k\Omega}$. The proof is finished. □

Remark 3.5. If the valued quiver $(\mathbf{M}, \Gamma, \Omega)$ is simply laced, then we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{ind}\mathcal{C}(\Gamma) & \xrightarrow[\text{(resp., } R(S_k^-))]{R(S_k^+)} & \operatorname{ind}\mathcal{C}(s_k\Gamma) \\ \Psi_\Gamma\big\downarrow & & \big\downarrow\Psi_{s_k\Gamma} \\ \operatorname{indrep}\widetilde{\Gamma} & \xrightarrow[\text{(resp., } \Sigma_k^-)]{\Sigma_k^+} & \operatorname{indrep}\widetilde{s_k\Gamma} \\ \mathbf{sdim}\big\downarrow & & \big\downarrow\mathbf{sdim} \\ \Phi_{\geq -1} & \xrightarrow{\sigma_k} & \Phi_{\geq -1}, \end{array}$$

and the Ω-compatibility degree of α and β defined above is the same as that defined in [15] (compare [4]). This implies Theorem 4.7 there. We remark that the functors Σ_k^+ and Σ_k^- defined in [15] are not equivalences.

The next result shows that the Ω-compatibility degree function on $\Phi_{\geq -1}$ is independent of the orientation Ω of Γ. It is the same as that defined in [11] on $\Phi_{\geq -1}$. This gives a unified form of compatibility degree in the language of quiver representations.

Theorem 3.6. *Let* $(\mathbf{M}, \Gamma, \Omega_0)$ *be an alternating valued quiver of Dynkin type. The* Ω_0*-compatibility degree function on* $\Phi_{\geq -1}$ *is the same as the compatibility degree function given by Fomin and Zelevinsky in* [10,11].

Proof. We have to verify that the Ω_0-compatible degree function satisfies conditions (3.3) and (3.4). For any two orientations on the same graph Γ, one can get an admissible sequence $i_1, \ldots, i_n$ such that $\Omega' = s_{i_n} \cdots s_{i_1} \Omega$, where i_k is the sink of $s_{i_{k-1}} \cdots s_{i_1} \Omega$. Then, from Theorem 3.4, we have $(\alpha \parallel \beta)_{\Omega} = (\sigma_{i_n} \cdots \sigma_{i_1}(\alpha) \parallel \sigma_{i_n} \cdots \sigma_{i_1}(\beta))_{\Omega'}$. It follows that $(\tau_\varepsilon \alpha \parallel \tau_\varepsilon \beta)_{\tau_\varepsilon(\Omega)} = (\alpha \parallel \beta)_{\Omega}$, for any $\alpha, \beta \in \Phi_{\geq -1}$, any $\varepsilon \in \{-1, 1\}$. This proves that (3.4) holds. Let $\beta \in \Phi_{\geq -1}$. Then $(-\alpha_i \parallel \beta)_{\Omega_0} = l({}_{\mathrm{End}\widetilde{P_i[1]}}\mathrm{Ext}^1_{\mathcal{C}(\Omega_0)}(\widetilde{P_i[1]}, \widetilde{M(\beta)})) = l({}_{\mathrm{End}\widetilde{P}_i}\mathrm{Hom}_{\mathcal{C}(\Omega_0)}(\widetilde{P}_i, \widetilde{M(\beta)}))$. It equals $l({}_{\mathrm{End}_H P_i}\mathrm{Hom}_H(P_i, M(\beta)))$ (this follows from Proposition 1.7 in [4]) and then it equals $[\beta : \alpha_i]$ if β is a positive root, or 0 otherwise. This proves that (3.3) holds. The proof is finished. □

This theorem extends Proposition 4.2 in [4] since in the simply laced case, the Γ-compatible degree defined in [15] is also the same as the compatible degree [11]. In view of Theorem 3.6, we denote $(\alpha \parallel \beta)_{\Omega_0}$ just by $(\alpha \parallel \beta)$.

Definition 3.7. A subset C of $\Phi_{\geq -1}$ is called compatible if $(\alpha \parallel \beta) = 0$ for all $\alpha, \beta \in C$. The subset C is called a cluster if it is maximal compatible.

Definition 3.8. The negative support $S(C)$ of a subset C of $\Phi_{\geq -1}$ is defined by $S(C) = \{i \in \Gamma : -\alpha_i \in C\}$. The subset C is called positive if $C \subset \Phi_{>0}$, i.e. $S(C) = \emptyset$ [11,15].

Combining Theorem 3.6 with Proposition 1.7 in [4], we re-prove Propositions 3.3, 3.5 and 3.6 of [11] in the language of quiver representations.

Proposition 3.9. *Let Γ be any Dynkin diagram, k a vertex of Γ and α, β almost positive roots. Then*

(1) $(\alpha \parallel \beta) = (\beta \parallel \alpha)$ *if Γ is a simply laced Dynkin quiver.*
(2) σ_k *sends a compatible subset to a compatible subset. In particular, it sends clusters to clusters.*
(3) *If α and β belong to $\Phi(J)_{\geq -1}$ for some proper subset $J \subset \Gamma$, then their compatibility degree with respect to the root subsystem $\Phi(J)$ is equal to $(\alpha \parallel \beta)$.*
(4) *If $\Gamma_1, \ldots, \Gamma_r \subset \Gamma$ are the connected components of the Coxeter graph, then the compatible subsets (resp., clusters) for $\Phi(\Gamma)_{\geq -1}$ are the disjoint unions $A_1 \amalg \cdots \amalg A_r$, where each A_k is a compatible subset (resp., cluster) for $\Phi(\Gamma_k)_{\geq -1}$.*
(5) *For every subset $J \subset \Gamma$, the correspondence $C \mapsto C - \{-\alpha_i : i \in J\}$ is a bijection between the set of all compatible subsets (resp., clusters) for $\Phi(\Gamma)_{\geq -1}$ with negative support J and the set of all positive compatible subsets (resp., clusters) for $\Phi(\Gamma - J)_{\geq -1}$.*

Proof. Let $(\mathbf{M}, \Gamma, \Omega)$ be a valued quiver with the underlying diagram Γ such that k is a sink. Statement (1) follows from $\mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M(\alpha), M(\beta)) = \mathrm{Ext}^1_{\mathcal{C}(\Omega)}(M(\beta), M(\alpha))$ [4]. For statement (2), we note that $(\sigma_k(\alpha) \parallel \sigma_k(\beta)) = (\sigma_k(\alpha) \parallel \sigma_k(\beta))_{s_k \Omega} = 0$ if and only if $(\alpha \parallel \beta)_{\Omega} = (\alpha \parallel \beta) = 0$. Statement (3) follows from Proposition 3.1. Statement (4) follows from (3), Proposition 3.1 and the obvious fact that $\mathrm{ind}\, \mathcal{C}(\Gamma) = \mathrm{ind}\, \mathcal{C}(\Gamma_1) \amalg \cdots \amalg \mathrm{ind}\, \mathcal{C}(\Gamma_r)$. For the proof of (5), we assume $C = \{\tilde{X} : X \in \mathrm{ind}\, H\} \amalg \{\tilde{P}_i[1] : i \in J\}$ is a compatible subset (resp., cluster) for $\Phi(\Gamma)_{\geq -1}$ with negative support J. Then $\mathrm{Ext}^1_{\mathcal{C}(H)}(\tilde{P}_i[1], \tilde{X}) = 0$. It follows that

$$\begin{aligned} \mathrm{Hom}_H(P_i, X) &= \mathrm{Hom}_{\mathcal{C}(H)}(\tilde{P}_i, \tilde{X}) \\ &= \mathrm{Hom}_{\mathcal{C}(H)}(\tilde{P}_i[1], \tilde{X}[1]) = \mathrm{Ext}^1_{\mathcal{C}(H)}(\tilde{P}_i[1], \tilde{X}) \\ &= 0. \end{aligned}$$

Then $X \in \mathrm{ind}\, A$ where $A = H/I$ is the quotient algebra of H whose modules are exactly the H-modules without composition factors E_i, with $i \in J$ (compare Proposition 3.1). Then $C - \{\tilde{P}_i[1] : i \in J\} = \{\tilde{X} : X \in \mathrm{ind}\, H\}$ is a compatible subset (resp., cluster) of $\Phi(\Gamma - J)_{\geq -1}$. Conversely, given a compatible subset (resp., cluster) $C_1 = \{\tilde{X} : X \in \mathrm{ind}\, A\}$ of $\Phi(\Gamma - J)_{\geq -1}$, $C_1 \amalg \{\tilde{P}_i[1] : i \in J\}$ is a compatible subset (resp., cluster) for $\Phi(\Gamma)_{\geq -1}$ with negative support J. The proof is finished. □

As a consequence of Theorems 3.4 and 3.6, we have the second main result of the paper which is a generalization of Theorem 4.5 in [4] and confirm positively Conjecture 9.1 for all Dynkin types.

Theorem 3.10. *Let $(\Gamma, \mathbf{d}, \Omega)$ be any Dynkin valued quiver, $\Phi_{\geq -1}$ the set of almost positive roots of the corresponding Lie algebra. Then the bijection $\gamma_\Omega : \mathrm{ind}\, \mathcal{C}(\Omega) \to \Phi_{\geq -1}$ induces a bijection between the following sets:*

(1) *The set of basic tilting objects in* $\mathcal{C}(\Omega)$.
(2) *The set of clusters in* $\Phi_{\geq -1}$.

Moreover, if we take the orientation Ω *to be the* Ω_0 *such that* (Γ, Ω_0) *is an alternating valued quiver, then the bijection* $\mathcal{P} \circ \gamma_{\Omega_0}$ *from* ind $\mathcal{C}(\Omega_0)$ *to the set of cluster variables of a cluster algebra of type* Γ *sends basic tilting objects in* $\mathcal{C}(\Omega_0)$ *to clusters of this cluster algebra, where* $\mathcal{P}$ *is the 1–1 correspondence from* $\Phi_{\geq -1}$ *to the set of cluster variables of* $\mathcal{A}$ *(Theorem 1.9* [10]*).*

Proof. The subset A of $\Phi_{\geq -1}$ is a cluster if and only if the subset $\gamma_{\Omega}^{-1}(A)$ of ind $\mathcal{C}(\Omega)$ is a basic tilting set. Combining this with Theorem 1.9 of [10], we finish the proof. □

By this theorem, we have that the tilting graph Δ_{Ω} is a realization of the exchange graph $E(\Phi)$ of [11]. Then Theorem 5.1 of [4] gives a quiver interpretation of Theorem 1.15 [11].

Corollary 3.11. *For every cluster* C *and every element* $\alpha \in C$*, there is a unique cluster* C' *such that* $C \cap C' = C - \alpha$*. Thus the exchange graph* $E(\Phi)$ *is regular of degree* n*: every vertex in* $E(\Phi)$ *is incident to precisely* n *edges.*

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