Cluster-tilted algebras and their intermediate coverings

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Abstract

The intermediate coverings of cluster-tilted algebras are constructed from the repetitive cluster categories which are defined in this paper. These repetitive cluster categories are Calabi-Yau triangulated categories with fractional CY-dimension and have also cluster tilting objects. Furthermore we show that the representations of these intermediate coverings of cluster-tilted algebras are induced from the repetitive cluster categories.

Key words. Repetitive cluster categories, cluster-tilted algebras, cluster tilting objects(subcategories), coverings.

Mathematics Subject Classification. 16G20, 16G70.

1 Introduction

Cluster categories defined in [BMRRT], and in [CCS] for type $A_n$, are the orbit categories $\frac{D^b(H)}{\tau^\mathbb{Z}}$ of derived categories $D^b(H)$ of a hereditary algebra $H$ by the automorphism group generated by $F = \tau^{-1}[1]$, where $\tau$ is the Auslander-Reiten translation in $D^b(H)$ and $[1]$ is the shift functor of $D^b(H)$. They are triangulated categories and are Calabi-Yau categories of CY-dimension 2 [K1].

Cluster-tilted algebras defined in [BMRRT][BMR1] are by definition, the endomorphism algebras of cluster tilting objects in the cluster categories of hereditary algebras. Together with cluster categories they provide an algebraic understanding (see [BMRRT] [CK1] [CK2]) of combinatorics of cluster algebras defined and studied by Fomin and Zelevinsky in [FZ]. In this connection, the indecomposable exceptional objects in cluster categories correspond to the cluster variables, and cluster tilting objects(= maximal 1–orthogonal subcategories [I1, I2]) to clusters of corresponding cluster algebras, see [CK1, CK2]. We refer to [K2] for a more recent nice survey on this topic.

It was proved in [KR] that cluster-tilted algebras provide a class of Gorenstein algebras of Gorenstein dimension at most 1, which are important in representation theory of algebras [Rin2]. Since they were introduced, they have been studied by many authors, see for example: [ABS1-3], [BFPPT], [BKL], [BM], [BMR1-2], [CCS], [KR], [KZ], [IY], [Rin2-4], [Zh]....

Now let $\mathcal{H}$ be a hereditary abelian category with tilting objects. The endomorphism category of a tilting object in $\mathcal{H}$ is called a quasi-tilted algebra [HRS]. The class of quasi-tilted algebras consists of tilted algebras and canonical algebras [H2]. From [H2], $\mathcal{H}$ is

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either derived equivalent to $\text{mod}\, H$ for a hereditary algebra $H$ or to the category $\text{coh}\, P$ of coherent sheaves over a weighted projective line $P$. The latter is derived equivalent to the module categories of canonical algebras [Rin1]. From such a hereditary abelian category $\mathcal{H}$, one can also define cluster category $\mathcal{C}(\mathcal{H})$ as the orbit category of $D^b(\mathcal{H})$ by $\tau^{-1}[1]$ [BMRRRT][Zh][BKL]. The cluster tilting objects in such a cluster category $\mathcal{C}(\mathcal{H})$ coincide with tilting objects in $\mathcal{H}$ by [BKL]. For a general hereditary category $\mathcal{H}$, it was shown that any cluster tilting object is induced from a tilting object of a hereditary abelian category which is derived equivalent to $\mathcal{H}$ [BMRRRT]. The endomorphism algebra of a cluster tilting object in $\mathcal{C}(\mathcal{H})$ is called a cluster-tilted algebra of type $\mathcal{H}$.

The aim of the note is to show that the coverings of cluster-tilted algebras can be constructed from repetitive cluster categories. Repetitive cluster categories are defined as the orbit categories of the derived categories $D^b(\mathcal{H})$ by the group $< F_m >$ generated by $F^m$, for any positive integer $m$. They are triangulated by Keller [K1], which are Calabi-Yau categories with fractional Calabi-Yau dimension. The cluster tilting objects in the repetitive cluster categories are shown to correspond bijectively to ones in the cluster categories; the endomorphism algebras of cluster tilting objects in $D^b(\mathcal{H})/ < F_m >$ are the coverings of the corresponding cluster-tilted algebras. They all share a Galois covering: the endomorphism algebra of the corresponding cluster tilting subcategory in $D^b(\mathcal{H})$.

This note is organized as follows:

In Section 2 we collect basic material on cluster tilting objects and cluster-tilted algebras. We generalize the Assem-Bristle-Schiffler’s characterization [ABS1] of cluster-tilted algebras to the general case: the cluster-tilted algebras of type $\mathcal{H}$, i.e., we show that the trivial extension algebra $A = B \ltimes Ext^2_B(DB, B)$ is a cluster-tilted algebra of type $\mathcal{H}$ if and only if $B$ is a quasi-tilted algebra.

In Section 3 we first introduce the repetitive cluster categories $\mathcal{C}_{F^m}(\mathcal{H})$, which are triangulated categories and are coverings of the corresponding cluster categories $\mathcal{C}(\mathcal{H})$. We then study cluster tilting theory in these triangulated categories.

2 Basics on cluster-tilted algebras

Let $\mathcal{D}$ be a $k$–linear triangulated category with finite dimensional Hom-spaces over a field $k$ and with Serre duality. We assume that $\mathcal{D}$ is a Krull-Remak-Schmidt category. Let $\mathcal{T}$ be a full subcategory of $\mathcal{D}$ closed under taking direct summands. The quotient category of
\( \mathcal{D} \) by \( \mathcal{T} \) denoted by \( \mathcal{D}/\mathcal{T} \), is by definition, a category with the same objects as \( \mathcal{D} \) and the space of morphisms from \( X \) to \( Y \) is the quotient of group of morphisms from \( X \) to \( Y \) in \( \mathcal{D} \) by the subgroup consisting of morphisms factor through an object in \( \mathcal{T} \). The quotient \( \mathcal{D}/\mathcal{T} \) is also an additive Krull-Remak-Schmidt category (see for example Lemma 2.1 in [KZ]). For \( X, Y \in \mathcal{D} \), we use \( \text{Hom}(X,Y) \) to denote \( \text{Hom}_\mathcal{D}(X,Y) \) for simplicity, and define that \( \text{Ext}^k(X,Y) := \text{Hom}(X,Y[k]) \). For a subcategory \( \mathcal{T} \), we say that \( \text{Ext}^i(\mathcal{T},\mathcal{T}) = 0 \) provided that \( \text{Ext}^i(X,Y) = 0 \) for any \( X, Y \in \mathcal{T} \). For an object \( T \), \text{add}T denotes the full subcategory consisting of direct summands of direct sum of finitely many copies of \( T \). Throughout the article, the composition of morphisms \( f : M \to N \) and \( g : N \to L \) is denoted by \( fg : M \to L \). For basic references on representation theory of algebras and triangulated categories, we refer [Rin1] and [H1].

Fix a triangulated category \( \mathcal{D} \), and assume that \( \mathcal{T} \) is a functorially finite subcategory of \( \mathcal{D} \) (see for example [AS]).

**Definition 2.1.**

1. \( \mathcal{T} \) is called rigid, provided \( \text{Ext}^1(\mathcal{T},\mathcal{T}) = 0 \); in particular, an object \( T \) is called rigid provided \( \text{Ext}^1(T,T) = 0 \).
2. \( \mathcal{T} \) is called a cluster tilting subcategory provided \( X \in \mathcal{T} \) if and only if \( \text{Ext}^1(X,\mathcal{T}) = 0 \) and \( X \in \mathcal{T} \) if and only if \( \text{Ext}^1(\mathcal{T},X) = 0 \). An object \( T \) is a cluster tilting object if and only if \( \text{add}T \) is a cluster tilting subcategory.

**Remark 2.2.**

1. Not all triangulated categories have cluster tilting subcategories, see for example, the example in Section 5 in [KZ].
2. In a module category \( \Lambda - \text{mod} \) of a self-injective algebra \( \Lambda \), \( T \oplus \Lambda \) is a cluster tilting module (=maximal 2−orthogonal module in [I1, I2]) if and only if \( T \) is cluster tilting in \( \Lambda - \text{mod} \).

**Remark 2.3.**

It was proved in [KZ] that if \( \mathcal{T} \) is contravariantly finite and satisfies the condition that \( X \in \mathcal{T} \) if and only if \( \text{Ext}^1(\mathcal{T},X) = 0 \), then \( \mathcal{T} \) is a cluster tilting subcategory.

For a triangulated category \( \mathcal{D} \) with Serre duality \( \Sigma \), \( \mathcal{D} \) has Auslander-Reiten triangles and \( \Sigma = \tau[1] \), where \( \tau \) is the Auslander-Reiten translation. Denote by \( F = \tau^{-1}[1] \).

**Lemma 2.4.** Let \( \mathcal{D} \) be a triangulated category with Serre duality \( \Sigma \), and \( \mathcal{T} \) a cluster tilting subcategory of \( \mathcal{D} \). Then \( FT = T \).

**Proof.** The assertion was proved in [KZ] or [IY].

The following results were proved in [KZ], see also [BMRRT][BMR1][KR][IY].

**Theorem 2.5.** Let \( T \) be a cluster tilting object of a triangulated category \( \mathcal{D} \), and \( A = \text{End}_\mathcal{D}T \). Then the following hold:
1. (Corollaries 4.4, 4.5 in [KZ]). The functor Hom(T, -) : D → A – mod induces an equivalence D/\text{add}(T[1]) \cong A – mod, and A is a Gorenstein algebra of Gorenstein dimension at most 1.

2. (Proposition 4.8 in [KZ]). Assume that the field k is algebraically closed. If B = End_D T' is the endomorphism algebra of another cluster tilting object T', then A and B have same representation type.

Let T = T_1 ⊕ T' be a cluster tilting object of a triangulated category D, where T_1 is indecomposable object. Let T_1 → E \xrightarrow{f} T_1 → T_1'[1] be the triangle with f a minimal right \text{add}T'–approximation of T_1. It follows from [IY] that T* = T_1 * ⊕ T' is a cluster tilting object and there is a triangle T_1 → E \xrightarrow{g} T_1 → T_1[1] with g being a minimal right \text{add}T'–approximation of T_1'. Let A, B be the endomorphism algebras of cluster tilting objects T, T* respectively. Denote by S_{T_1}, (or S_{T'}) the simple A-module corresponding to T_1 (resp. simple B-module corresponding to T_1'). The following proposition is a generalization of Proposition 2.2 in [KR] and Theorem B in [BMR1].

**Proposition 2.6.** Let T and T* be as above. Then A – mod/\text{add}S_{T_1} \cong B – mod/\text{add}S_{T'}.

**Proof.** Denote by G = Hom(T, -). The induced functor \( \overline{G} : D/\text{add}(T[1]) \rightarrow A – \text{mod} \) is an equivalence by Theorem 2.5(1). We consider the composition of the functor \( \overline{G} \) with the quotient functor Q : A – mod → \text{add}(\text{Hom}(T, T_1'[1])^\text{A–mod}) which is denoted by G_1. The functor G_1 is full and dense since \( \overline{G} \) and Q are. Under the equivalence \( \overline{G}, T_1'[1] \) corresponds to Hom(T, T_1'[1]). For any morphism \( \varphi : X \rightarrow Y \) in the category \text{add}(\text{Hom}(T, T_1'[1])) \approx S_{T_1} \) and Hom(T, T_1'[1]) \approx S_{T'} \) (compare Lemma 4.1 in [BMR1]). Then \( A – \text{mod}/\text{add}S_{T_1} \approx B – \text{mod}/\text{add}S_{T'} \). The proof is finished.

From now on, we assume that \( \mathcal{H} \) is a hereditary \( k \)-linear category with finite dimensional Hom-spaces and Ext-spaces. We also assume that \( \mathcal{H} \) has tilting objects. The endomorphism algebra of tilting object \( T \) in \( \mathcal{H} \) is called a quasi-tilted algebra [HRS]. Since \( \mathcal{H} \) has tilting objects, \( D^b(\mathcal{H}) \) has Serre duality [HRS], and has also Auslander-Reiten triangles, the Auslander-Reiten translation is denoted by \( \tau \) [HRS]. Let \( F = \tau^{-1}[1] \) be the automorphism of the bounded derived category \( D^b(\mathcal{H}) \). We call the orbit category \( D^b(\mathcal{H})/\text{< F }> \) the cluster category of type \( \mathcal{H} \), which is denoted by \( \mathcal{C}(\mathcal{H}) \) [BMRRT]. For cluster tilting theory in the cluster category \( \mathcal{C}(\mathcal{H}) \), we refer [BKL][BMRRT][Zh]. The endomorphism algebra \( \text{End}_{\mathcal{C}(\mathcal{H})} T \) of a cluster tilting object \( T \) in \( \mathcal{C}(\mathcal{H}) \) is called a cluster-tilted algebra of type \( \mathcal{H} \). When \( \mathcal{H} \) is the module category over a hereditary algebra \( H = kQ \), we call the corresponding orbit category the cluster category of \( H \) or of \( Q \). In this case the endomorphism algebra of a cluster tilting object is called a cluster-tilted algebra of \( H \) [BM], [BMR1], [Zh], [ABS1-3].
Now we give a characterization of cluster-tilted algebras of type $\mathcal{H}$, which generalizes some results in [ABS1], [Zh].

Given any finite-dimensional algebra $B$, from the $B$–bimodule $\text{Ext}^2(DB, B)$, one can form the trivial extension algebra of $B$ with the bimodule $\text{Ext}^2(DB, B)$: $A = B \ltimes \text{Ext}^2(DB, B)$. It was proved that this trivial extension algebra is a cluster-tilted algebra of $H$ if and only if $B$ is a tilted algebra [ABS1], see also [Zh]. In the following, we generalize the characterization of cluster-tilted algebras to the cluster-tilted algebras of type $\mathcal{H}$. The proof is exactly the same as the proof in [ABS1], we omit it here.

**Proposition 2.7.** Let $A = B \ltimes \text{Ext}^2(DB, B)$. Then $A$ is a cluster-tilted algebra of type $\mathcal{H}$ if and only if $B$ is a quasi-tilted algebra, i.e. the endomorphism algebra of a tilting object in $\mathcal{H}$.

### 3 Intermediate covers of cluster tilted algebras of type $\mathcal{H}$

As in the previous section, $\mathcal{H}$ denotes a hereditary $k$–linear category with finite dimensional Hom-spaces and Ext-spaces. We assume that $\mathcal{H}$ has tilting objects. Since $\mathcal{H}$ has tilting objects, $D^b(\mathcal{H})$ has Serre duality, and also Auslander-Reiten translate $\tau$ (AR-translate for short)[HRS]. Let $F = \tau^{-1}[1]$ be the automorphism of the bounded derived category $D^b(\mathcal{H})$. Fix a positive integer $m$ throughout this section. We consider the orbit category $D^b(\mathcal{H})/ < F^m >$, which is by definition a $k$–linear category whose objects are the same in $D^b(\mathcal{H})$, and whose morphisms are given by:

$$\text{Hom}_{D^b(\mathcal{H})/ < F^m >} (\tilde{X}, \tilde{Y}) = \oplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\mathcal{H})} (X, (F^m)^i Y).$$

Here $X$ and $Y$ are objects in $D^b(\mathcal{H})$, and $\tilde{X}$ and $\tilde{Y}$ are the corresponding objects in $D^b(\mathcal{H})/ < F^m >$ (although we shall sometimes write such objects simply as $X$ and $Y$).

**Definition 3.1.** The orbit category $D^b(\mathcal{H})/ < F^m >$ is called the repetitive cluster category of type $\mathcal{H}$. We denote it by $\mathcal{C}_{F^m}(\mathcal{H})$.

**Remark 3.2.** When $m = 1$, we get back to the usual cluster category $\mathcal{C}(\mathcal{H})$, which was introduced by Buan-Marsh-Reineke-Reiten-Todorov in [BMRRT], and also by Caldero-Chapoton-Schiffler in [CCS] for $A_n$ case.

The repetitive cluster categories $\mathcal{C}_{F^m}(\mathcal{H})$ serve as intermediate categories between the cluster categories $\mathcal{C}(\mathcal{H})$ and derived categories $D^b(\mathcal{H})$. Similarly as for the case of cluster categories, for any positive integer $m$, we have a natural projection functor $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$. If $m = 1$, the projection functor $\pi_m$ is simply denoted by $\pi$.

Now we define a functor $\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$, which sends objects $\tilde{X}$ in $\mathcal{C}_{F^m}(\mathcal{H})$ to objects $\tilde{X}$ in $\mathcal{C}(\mathcal{H})$ and morphisms $f : \tilde{X} \to \tilde{Y}$ in $\mathcal{C}_{F^m}(\mathcal{H})$ to the morphisms $f : \tilde{X} \to \tilde{Y}$ in $\mathcal{C}(\mathcal{H})$.

It is easy to check that $\pi = \rho_m \circ \pi_m$.

One can identify the set $\text{ind}\mathcal{C}(\mathcal{H})$ with the fundamental domain for the action of $F$ on $\text{ind}D^b(\mathcal{H})$ [BMRRT]. Passing to the orbit category $\mathcal{C}_{F^m}(\mathcal{H})$, one can view $\text{ind}\mathcal{C}(\mathcal{H})$ as a (usually not full) subcategory of $\text{ind}\mathcal{C}_{F^m}(\mathcal{H})$.
Proposition 3.3. 1. $\mathcal{C}_{Fm}(\mathcal{H})$ is a triangulated category with Auslander-Reiten triangles and Serre functor $\Sigma = \tau[1]$, where $\tau$ is the AR-translate in $\mathcal{C}_{Fm}(\mathcal{H})$, which is induced from AR-translate in $D^b(\mathcal{H})$.

2. The projections $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{Fm}(\mathcal{H})$ and $\rho_m : \mathcal{C}_{Fm}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ are triangle functors and also covering functors.

3. $\mathcal{C}_{Fm}(\mathcal{H})$ is a fractional Calabi-Yau category of CY-dimension $\frac{2m}{m}$.

4. $\mathcal{C}_{Fm}(\mathcal{H})$ is a Krull-Remak-Schmidt category.

5. $\text{ind}\mathcal{C}_{Fm}(\mathcal{H}) = \bigcup_{i=0}^{m-1}(\text{ind}\mathcal{C}(\mathcal{H}))$.

Proof. 1. It follows from [K1] that $\mathcal{C}_{Fm}(\mathcal{H})$ is a triangulated category. The remaining claims follow from Proposition 1.3 [BMRRT].

2. It is proved in Corollary 1 in Section 8.4 of [K1] that $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{Fm}(\mathcal{H})$ is a triangle functor. It is easy to check that $\pi \circ F^m \cong \pi$. By the universal property of the orbit category $D^b(\mathcal{H})/ < F^m > [K1]$, we obtain a triangle functor $\rho : \mathcal{C}_{Fm}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ satisfying that $\rho \pi_m = \pi$, which turns out to be the functor $\rho_m$.

3. The Serre functor $\Sigma = \tau[1]$ in $\mathcal{C}_{Fm}(\mathcal{H})$ satisfies that $\Sigma^m = \tau^m[m] = F^m[2m] \cong [2m]$. Therefore $\mathcal{C}_{Fm}(\mathcal{H})$ is a fractional Calabi-Yau category with CY-dimension $\frac{2m}{m}$.

4. The proof given in Proposition 1.6 [BMRRT] for $m = 1$, can be modified to work for any positive value of $m$.

We note that if the hereditary abelian category $\mathcal{H}$ is equivalent to the module category of a finite dimensional hereditary algebra $H$, then the indecomposable objects in $\mathcal{C}(\mathcal{H})$ are of form $\tilde{M}$ or of form $P[1]$, where $M$ is an indecomposable $H$-module and $P[1]$ is the first shift of an indecomposable projective $H$-module $P$. If the hereditary abelian category $\mathcal{H}$ is not equivalent to the module category of a finite dimensional hereditary algebra $H$, then the indecomposable objects in $\mathcal{C}(\mathcal{H})$ are of form $\tilde{M}$, where $M$ is an indecomposable object in $\mathcal{H}$.

Now we discuss the cluster tilting objects in $\mathcal{C}_{Fm}(\mathcal{H})$. Denoted by $F = \tau^{-1}[1]$, which can be viewed as an automorphism of $D^b(\mathcal{H})$ or of $\mathcal{C}_{Fm}(\mathcal{H})$. The following proposition is a generalization of Lemma 4.14 in [KZ].

Proposition 3.4. An object $T$ in $\mathcal{C}_{Fm}(\mathcal{H})$ is a cluster tilting object if and only if $\pi_m^{-1}(\text{add}\ T)$ is a cluster tilting subcategory of $D^b(\mathcal{H})$.

Proof. We only give a detailed proof in the case $\mathcal{H}$ is equivalent to the module category of a finite dimensional hereditary algebra $H$. The proof in case $\mathcal{H}$ is not of the form is similar.

Suppose that $\mathcal{H} \approx H - \text{mod}$, where $H$ is a finite dimensional hereditary algebra over a field $k$. For an object $T$ in $\mathcal{C}_{Fm}(\mathcal{H})$, we denote $T = \pi_m^{-1}(\text{add}\ T)$, which is a full subcategory of $D^b(\mathcal{H})$. It is easy to prove that $F(T) = T$ in $D^b(\mathcal{H})$ if and only if $F(\text{add}\ T) = \text{add}\ T$ in $\mathcal{C}_{Fm}(\mathcal{H})$. 


Suppose \( T \) is a cluster tilting subcategory of \( D^b(H) \). Then \( FT = T \) by Lemma 2.4 or Proposition 4.7 [KZ]. Hence \( F(\text{add} T) = \text{add} T \) in \( C_{Fm}(H) \). We denote by \( T' \) the intersection of \( T \) with the additive subcategory \( C' \) generated by all \( H \)-modules as stalk complexes of degree 0 together with \( H[1] \). Then we have that \( T = \{ F^n(T') \mid n \in \mathbb{Z} \} \). Now \( \pi_m(T) = \pi_m(\bigcup_{i=0}^{m-1} F^i(T')) \), denoted by \( T_1 \).  For any pair of objects \( T_1, T_2 \in T_1 \), there are \( T_1, T_2 \in T' \) such that \( T_1 = F^t(\pi_m(T_1)), T_2 = F^s(\pi_m(T_2)) \) with \( 0 \leq t, s \leq m - 1 \). Then \( \text{Ext}^1(T_1, T_2) = \text{Hom}(T_1, T_2[1]) \cong \oplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(H)}(F^n(T_1), (F^n)^*F^t(T_2[1])) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) \). By an easy computation, one has that \( \text{Hom}_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = 0 \) if \( nm + t - s \leq -2 \) or \( nm + t - s \geq 1 \). When \( nm + t - s = -1 \), \( \text{Hom}_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = \text{Hom}_{D^b(H)}(T_1, F^{-1}T_2[1]) = \text{Hom}_{D^b(H)}(T_1, T_2) \cong \text{Ext}^1_{D^b(H)}(T_2, T_1) \), which equals 0 by the fact that \( T \) is a cluster tilting subcategory of \( D^b(H) \). When \( nm + t - s = 0 \), \( \text{Hom}_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = \text{Hom}_{D^b(H)}(T_1, T_2[1]) = \text{Ext}^1_{D^b(H)}(T_1, T_2) \), which equals 0 by the fact that \( T \) is a cluster tilting subcategory of \( D^b(H) \). Therefore \( \text{Ext}^1(T_1, T_2) = 0 \), i.e. \( T_1 \) is rigid in \( C_{Fm}(H) \).

If there are indecomposable objects \( X = \pi_m(X) \in C_{Fm}(H) \) with \( X \in D^b(H) \) satisfying \( \text{Ext}^1(T_1, X) = 0 \), then \( \text{Ext}^1(F^nT', X) = 0 \) for any \( n \), and then \( \text{Ext}^1(T, X) = 0 \). Hence \( X \in T \) since \( T \) is a cluster tilting subcategory. Thus \( X \in T_1 \). This proves that the image \( T_1 \) of \( T \) under \( \pi_m \) is a cluster tilting subcategory of \( C_{Fm}(H) \).

Conversely, from \( T = \pi_m^{-1}(T_1) \) and \( F(T_1) = T_1 \), we get \( F(T) = T \). As above we denote by \( T' \) the intersection of \( T \) with the additive subcategory \( C' \) generated by all \( H \)-modules as stalk complexes of degree 0 together with \( H[1] \). Then \( T = \{ F^n(T') \mid n \in \mathbb{Z} \} \) and \( T_1 = \pi_m(T) = \pi_m(\bigcup_{i=0}^{m-1} F^i(T')) \). From \( T_1 \) being contravariantly finite, we have \( T \) is also contravariantly finite. Since \( \text{Ext}^1(T_1, T_1) \cong \oplus_{n \in \mathbb{Z}} \text{Ext}^1_{D^b(H)}(\bigcup_{i=0}^{m-1} F^i(T'), F^n(\bigcup_{i=0}^{m-1} F^i(T'))) = 0 \), we have that \( \text{Ext}^1_{D^b(H)}(F^nT', F^nT') \cong \text{Ext}^1_{D^b(H)}(T', F^n-mT') = 0 \). This proves that \( T \) is a rigid subcategory. Now if \( X \in D^b(H) \) satisfies \( \text{Ext}^1_{D^b(H)}(T, X) = 0 \), then \( \text{Ext}^1_{C_{Fm}(H)}(F^n(T_1), X) = 0 \), \( \forall 0 \leq i \leq m - 1 \). It follows that \( X \in T_1 \), hence \( X \in T \). Similarly, if \( X \in D^b(H) \) satisfies \( \text{Ext}^1_{D^b(H)}(X, T) = 0 \), then \( X \in T \). 

From Proposition 3.4 above and Lemma 4.14 in [KZ], we have a one-to-one correspondence between the three sets: the set of cluster tilting subcategories in \( D^b(H) \); the set of cluster tilting subcategories in \( C_{Fm}(H) \); the set of cluster tilting subcategories in \( C(H) \), via triangle covering functors: \( \pi_m : D^b(H) \to C_{Fm}(H) \) and \( \rho_m : C_{Fm}(H) \to C(H) \).

**Theorem 3.5.** Let \( H \) be a hereditary abelian category with tilting objects. Let \( T \in C(H) \).

1. \( T \) is a cluster tilting object in cluster category \( C(H) \) if and only if \( \rho_m^{-1}(T) \) is a cluster tilting object in \( C_{Fm}(H) \) if and only if \( \pi^{-1}(\text{add} T) \) is a cluster tilting subcategory in \( D^b(H) \).

2. For any tilting object \( T' \in H, \oplus_{i=0}^{m-1} F^iT' \) is a cluster tilting object in \( C_{Fm}(H) \), and any cluster tilting object in \( C_{Fm}(H) \) arises in this way, i.e. there is a hereditary abelian category \( H' \), which is derived equivalent to \( H \), and a tilting object \( T \in H' \) such that the cluster tilting object is induced from \( T \).

**Proof.** 1. It follows Lemma 4.14 in [KZ] or the special case of Proposition 3.4 where \( m = 1 \), that \( T \) is a cluster tilting object in \( C(H) \) if and only if \( \pi^{-1}(\text{add} T) \) is a
cluster tilting subcategory in $D^b(\mathcal{H})$. By Proposition 3.4, we have that $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ if and only if $\pi_m^{-1}(\text{add}(\rho_m^{-1}(T)))$ is a cluster tilting subcategory in $D^b(\mathcal{H})$. Since $\pi = \rho_m \pi_m$, $\pi(\pi_m^{-1}(\text{add}(\rho_m^{-1}(T)))) = T$, we have that $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ if and only if $T$ is a cluster tilting object in $\mathcal{C}(\mathcal{H})$.

2. For any tilting object $T'$ in $\mathcal{H}$, from [BMRRRT] and [Zh], $T'$ is a cluster tilting object in $\mathcal{C}(\mathcal{H})$. Hence $\text{add}_{i=0}^{m-1}F^iT'$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ by the first part of the theorem. Suppose $M$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$. Then by the first part of the theorem, $\rho_m(M)$ is a cluster tilting object in the cluster category $\mathcal{C}(\mathcal{H})$. Therefore $\rho_m(M)$ is induced from a tilting object of a hereditary abelian category $\mathcal{H}'$, which is derived equivalent to $\mathcal{H}$ [Zh, BMRRRT]. Then $M$ is induced from a tilting object of $\mathcal{H}'$.

\[\square\]

**Definition 3.6.** We call the endomorphism algebras $\text{End}_{\mathcal{C}_{F^m}(\mathcal{H})}T$ of cluster tilting objects $T$ in the repetitive cluster category $\mathcal{C}_{F^m}(\mathcal{H})$ the generalized cluster-tilted algebras of type $\mathcal{H}$, or simply the generalized cluster-tilted algebras.

Now we study the representation theory of generalized cluster-tilted algebras. We recall that $\pi_m : D^b(\mathcal{H}) \longrightarrow \mathcal{C}_{F^m}(\mathcal{H})$ is the projection.

**Theorem 3.7.** Let $T$ be a tilting object in $\mathcal{H}$, $\tilde{A} = \text{End}_{\mathcal{C}_{F^m}(\mathcal{H})}(\text{add}_{i=0}^{m-1}F^iT)$ the generalized cluster-tilted algebra.

1. $\tilde{A}$ has a Galois covering $\pi_m : \pi^{-1}(\text{add}T) \rightarrow \rho_m^{-1}(\text{add}T)$ which is the restriction of the projection $\pi_m : D^b(\mathcal{H}) \rightarrow \mathcal{C}_{F^m}(\mathcal{H})$.

2. The projection $\pi_m$ induces a push-down functor $\tilde{\pi}_m : \frac{D^b(\mathcal{H})}{\text{add}_{i=0}^{m-1}F^iT} \longrightarrow \tilde{A} \mod$. Under this equivalence, the subcategory $\text{add}(\pi_m(T))$ corresponds to the subcategory of projective $\tilde{A}$–modules.

3. If $T'$ is a tilting object in $\mathcal{H}$, then the generalized cluster tilted algebra $\tilde{A}' = \text{End}_{\mathcal{C}_{F^m}(\mathcal{H})}(\text{add}_{i=0}^{m-1}F^iT')$ has the same representation type as $A$.

**Proof.** (1). Let $T = \text{add} (\{ F^i(T) | i \in \mathbb{Z} \})$, $T = \pi^{-1}(\text{add}T)$ is a cluster tilting subcategory of $D^b(\mathcal{H})$. Hence by Proposition 3.4, $\pi_m(T) = \rho_m^{-1}(\text{add}T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$. By Theorem 2.5, we have the equivalence $\mathcal{C}_{F^m}(\mathcal{H})/	ext{add}_{i=0}^{m-1}\pi_m(F^i(T)), -) : \mathcal{C}_{F^m}(\mathcal{H})/\text{add}_{i=0}^{m-1}\pi_m(F^i(T))) \rightarrow \tilde{A} \mod$. Under this equivalence, the subcategory $\text{add}(\pi_m(T))$ corresponds to the subcategory of projective $\tilde{A}$–modules.

The projection $\pi_m$ sends $T$ to $\pi_m(T)$. Thus $\pi_m|T : T \longrightarrow \rho_m^{-1}(T)$ is a Galois covering with Galois group generated by $F^m$.

(2). By Theorem 3.3 and Corollary 4.4 in [KZ] there are equivalences $D^b(\mathcal{H})/\text{add}T[1] \cong \text{mod}(T)$ and $\mathcal{C}_{m}(\mathcal{H})/(\pi_m(T)[1]) \cong \text{mod}(\pi_m(T))$. We define the induced functor $\tilde{\pi}_m$ as follows: $\tilde{\pi}_m(X) := \pi_m(X)$ for any object $X \in D^b(\mathcal{H})/\text{add}T[1]$, and $\tilde{\pi}_m(f) := \pi_m(f)$ for any morphism $f : X \rightarrow Y$ in $D^b(\mathcal{H})/\text{add}T$. Clearly $\tilde{\pi}_m$ is well-defined and makes the following diagram commutative:
\[ D^b(\mathcal{H}) \xrightarrow{\pi_m} \mathcal{C}(\mathcal{H}) \]
\[ P_1 \downarrow \]
\[ D^b(\mathcal{H})/T[1] \xrightarrow{\pi_m} \mathcal{C}(\mathcal{H})/\pi(T)[1]. \]

Where \( P_1, P_2 \) are the natural quotient functors. Then \( \pi_m \) is a covering functor from \( D^b(\mathcal{H})/T[1] \) to \( \mathcal{C}_{F^m}/\pi(T[1]) \), i.e., it is a covering functor from \( D^b(\mathcal{H})/T[1] \) to \( \tilde{A} - \text{mod} \) (\( \approx \text{mod}(\pi_m(T)) \)).

(3) This is a direct consequence of Theorem 2.2 \( \square \)

Similarly as above, the triangle covering functor \( \rho_m : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \) induces a covering functor from \( \tilde{A} \) to the cluster-tilted algebra \( \text{End}_{\mathcal{C}(\mathcal{H})}T \) indicated as the following Theorem.

**Theorem 3.8.** Let \( T \) be a tilting object in \( \mathcal{H} \), \( A = \text{End}_{\mathcal{C}(\mathcal{H})}T \) and \( \tilde{A} = \text{End}_{\mathcal{C}_{F^m}(\mathcal{H})}(\oplus_{i=0}^{m-1} F^i T) \) the generalized cluster-tilted algebra.

1. \( \rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \) restricted to the cluster tilting subcategory \( \text{add}(\bigcup_{i=0}^{m-1} F^i T) \) induces a Galois covering of \( A \).

2. The functor \( \rho_m \) also induces a push-down functor \( \tilde{\rho}_m : \tilde{A} - \text{mod} \longrightarrow A - \text{mod} \).

**Proof.** The strategy of the proof is almost the same as that of Theorem 3.7, we present it here for the convenience of reader.

(1) By Theorem 3.5, \( \rho_m^{-1}(T) \) is a cluster tilting object in \( \mathcal{C}_{F^m}(\mathcal{C}) \), and \( \rho_m^{-1}(T) = \oplus_{i=1}^{m-1} F^i(T) \). By Theorem 2.5, we have the equivalence \( \text{Hom}_{\mathcal{C}_{F^m}(\mathcal{H})}(\rho_m^{-1}(T), -) : \mathcal{C}_{F^m}(\mathcal{H}) \to \tilde{A} - \text{mod} \). Under this equivalence, the subcategory \( \text{add}(\rho_m^{-1}(T)) \) corresponds to the subcategory of projective \( \tilde{A} \)-modules.

The triangle functor \( \rho_m \) sends \( \text{add}(\rho_m^{-1}(T)) \) to \( \text{add}T \). Thus \( \rho_m|_{\text{add}(\rho_m^{-1}(T))} : \text{add}(\rho_m^{-1}(T)) \to \text{add}T \) is a Galois covering with Galois group \( Z_m \).

(2) By Theorem 3.3 and Corollary 4.4 in [KZ], there is an equivalence \( \mathcal{C}_m(\mathcal{H})/\langle \text{add}(\rho_m^{-1}(T))[1] \rangle \cong \tilde{A} - \text{mod} \). We define the induced functor \( \tilde{\rho}_m \) as follows: \( \tilde{\rho}_m(X) := \rho_m(X) \) for any object \( X \in \mathcal{C}_m(\mathcal{H})/\langle \rho_m^{-1}(T) \rangle[1] \), and \( \tilde{\rho}_m(f) := \rho_m(f) \) for any morphism \( f : X \to Y \) in \( \mathcal{C}_m(\mathcal{H})/\langle \rho_m^{-1}(T) \rangle \). Clearly \( \tilde{\rho}_m \) is well-defined and makes the following diagram commutative:

\[ \begin{array}{ccc} \mathcal{C}_{F^m}(\mathcal{H}) & \xrightarrow{\rho_m} & \mathcal{C}(\mathcal{H}) \\ P_1 \downarrow & & P_2 \downarrow \\ \mathcal{C}_{F^m}(\mathcal{H})/\text{add}(\rho_m^{-1}(T))[1] & \xrightarrow{\tilde{\rho}_m} & \mathcal{C}(\mathcal{H})/\text{add}(T[1]). \end{array} \]

Where \( P_1, P_2 \) are the natural quotient functors. Then \( \tilde{\rho}_m \) is a covering functor from \( \mathcal{C}_m(\mathcal{H})/\langle \text{add}(\rho_m^{-1}(T))[1] \rangle \) to \( \mathcal{C}/\text{add}(T[1]) \), i.e, it is a covering functor from \( \tilde{A} - \text{mod} \) to \( A - \text{mod} \). \( \square \)

**Remark 3.9.** By Proposition 2.7, see also [ABS,Zh], the cluster-tilted algebra \( A \) of type \( \mathcal{H} \) can be written as a trivial extension \( A = B \times M \), where \( M = \text{Ext}^2_B(DB,B) \). Then
A has as \( \mathbb{Z} \)-covering the following (infinite dimensional) matrix algebra (i.e. the cluster repetitive algebra in [ABS³]):

\[
A_\infty = \begin{bmatrix}
\vdots & B \\
\vdots & M & B \\
& \vdots & M & B \\
\end{bmatrix}
\]

On the other hand, \( A = B \times M \) is also a \( \mathbb{Z}_m \)-graded algebra. Then \( A \) has a \( \mathbb{Z}_m \)-covering \( A^\# \mathbb{Z}_m \), the smash product of graded algebra \( A \) with group \( \mathbb{Z}_m \) [CM].

**Examples**

1. Let \( D^b(H) \) be the (bounded) derived category of hereditary algebra \( H \), where \( H \) is the path algebra of the quiver:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\]

If we take \( T \) to be the subcategory generated by \( \{ \tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n \in \mathbb{Z} \} \), then \( T \) is a cluster tilting subcategory of \( D^b(H) \) and \( D^b(H)/T \cong A_\infty - \text{mod} \) where \( A_\infty \) is the algebra with quiver

\[
A_\infty: \cdots \circ \rightarrow \circ \rightarrow \circ \rightarrow \cdots
\]

and with \( \text{rad}^2 = 0 \) [KZ].

2. Let \( m = 1 \). We consider the cluster category \( \mathcal{C}(H) \). If we take \( T = P_a \oplus P_c \oplus S_a \), then \( T \) is a cluster tilting object of \( \mathcal{C}(H) \) and \( \mathcal{C}(H)/(\text{add}T) \cong A - \text{mod} \) where \( A \) is the algebra with quiver

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\]

and with \( \text{rad}^2 = 0 \).
3. Let $m = 2$. We consider the repetitive cluster category $C_{F^2}(A)$. If we take $T'$ to be the subcategory generated by $\{\tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n = 0, 1\}$, then $T'$ is a cluster tilting subcategory of $C_{F^2}(A)$ and $C_{F^2}(A)/T' \cong A_1 - \text{mod}$ where $A_1$ is the algebra with quiver

$$Q_1 : \quad \begin{array}{c}
\circ \quad \rightarrow \quad \circ \\
\uparrow \quad \circ \quad \rightarrow \quad \circ \\
\circ \quad \leftarrow \quad \circ \quad \leftarrow \quad \circ
\end{array}$$

and with $\text{rad}^2 = 0$.

4. Let $m = 3$. We consider the repetitive cluster category $C_{F^3}(A)$. If we take $T''$ to be the subcategory generated by $\{\tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n = 0, 1, 2\}$, then $T''$ is a cluster tilting subcategory of $C_{F^3}(A)$ and $C_{F^3}(A)/T'' \cong A_2 - \text{mod}$ where $A_2$ is the algebra with quiver

$$Q_2 : \quad \begin{array}{c}
\circ \quad \rightarrow \quad \circ \quad \rightarrow \quad \circ \quad \rightarrow \quad \circ \\
\circ \quad \leftarrow \quad \circ \quad \leftarrow \quad \circ \quad \leftarrow \quad \circ
\end{array}$$

and with $\text{rad}^2 = 0$.

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References


