Cluster-tilted algebras and their intermediate coverings

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Abstract

The intermediate coverings of cluster-tilted algebras are constructed from the repetitive cluster categories which are defined in this paper. These repetitive cluster categories are Calabi-Yau triangulated categories with fractional CY-dimension and have also cluster tilting objects. Furthermore we show that the representations of these intermediate coverings of cluster-tilted algebras are induced from the repetitive cluster categories.

Key words. Repetitive cluster categories, cluster-tilted algebras, cluster tilting objects(subcategories), coverings.

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1 Introduction

Cluster categories defined in [BMRRT], and in [CCS] for type A_n, are the orbit categories $\frac{D^b(H)}{<F>}$ of derived categories $D^b(H)$ of a hereditary algebra H by the automorphism group generated by $F = \tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $D^b(H)$ and [1] is the shift functor of $D^b(H)$. They are triangulated categories and are Calabi-Yau categories of CY-dimension 2 [K1].
Cluster-tilted algebras defined in [BMRRT][BMR1] are by definition, the endomorphism algebras of cluster tilting objects in the cluster categories of hereditary algebras. Together with cluster categories they provide an algebraic understanding (see [BMRRT] [CK1] [CK2]) of combinatorics of cluster algebras defined and studied by Fomin and Zelevinsky in [FZ]. In this connection, the indecomposable exceptional objects in cluster categories correspond to the cluster variables, and cluster tilting objects(= maximal $1-$orthogonal subcategories [I1, I2]) to clusters of corresponding cluster algebras, see [CK1, CK2]. We refer to [K2] for a more recent nice survey on this topic. It was proved in [KR] that cluster-tilted algebras provide a class of Gorenstein algebras of Gorenstein dimension at most 1, which are important in representation theory of algebras [Rin2]. Since they were introduced, they have been studied by many authors, see for example: [ABS1-3], [BFPPT], [BKL], [BM], [BMR1-2], [CCS], [KR], [KZ], [IY], [Rin2-4], [Zh]....

Now let $\mathcal{H}$ be a hereditary abelian category with tilting objects. The endomorphism algebra of a tilting object in $\mathcal{H}$ is called a quasi-tilted algebra [HRS]. The class of quasi-tilted algebras consists of tilted algebras and canonical algebras [H2]. From [H2], $\mathcal{H}$ is

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either derived equivalent to mod$H$ for a hereditary algebra H or to the category $cohP$ of coherent sheaves over a weighted projective line P. The latter is derived equivalent to the module categories of canonical algebras [Rin1]. From such a hereditary abelian category $\mathcal{H}$, one can also define cluster category $\mathcal{C}(\mathcal{H})$ as the orbit category of $D^b(\mathcal{H})$ by $\tau^{-1}[1]$ [BMRRT][Zh][BKL]. the cluster tilting objects in such a cluster category $\mathcal{C}(\mathcal{H})$ coincide with tilting objects in $\mathcal{H}$ by [BKL]. For a general hereditary category $\mathcal{H}$, it was shown that any cluster tilting object is induced from a tilting object of a hereditary abelian category which is derived equivalent to $\mathcal{H}$ [BMRRT]. The endomorphism algebra of a cluster tilting object in $\mathcal{C}(\mathcal{H})$ is called a cluster-tilted algebra of type $\mathcal{H}$. Since the ordinary quiver of a non-hereditary cluster-tilted algebra always has oriented cycles, it has non-trivial coverings [BoG][G]. A certain Galois covering of a cluster-tilted algebra was constructed by defining the cluster repetitive algebra of a tilted algebra in [ABS3], see also [KZ] for a different construction. For the notions of covering functors, and the notion of push-down functors, we refer to [BoG] [G].

The aim of the note is to show that the coverings of cluster-tilted algebras can be constructed from repetitive cluster categories. Repetitive cluster categories are defined as the orbit categories of the derived categories $D^b(\mathcal{H})$ by the group $< F^m >$ generated by F^m, for any positive integer m. They are triangulated by Keller [K1], which are Calabi-Yau categories with fractional Calabi-Yau dimension. The cluster tilting objects in the repetitive cluster categories are shown to correspond bijectively to ones in the cluster categories; the endomorphism algebras of cluster tilting objects in $D^b(\mathcal{H})/ < F^m >$ are the coverings of the corresponding cluster-tilted algebras. They all share a Galois covering: the endomorphism algebra of the corresponding cluster tilting subcategory in $D^b(\mathcal{H})$.
This note is organized as follows:
In Section 2 we collect basic material on cluster tilting objects and cluster-tilted algebras. We generalize the Assem-Brüstle-Schiffler's characterization [ABS1] of cluster-tilted algebras to the general case: the cluster-tilted algebras of type $\mathcal{H}$, i.e., we show that the trivial extension algebra $A = B \ltimes Ext^2_B(DB, B)$ is a cluster-tilted algebra of type $\mathcal{H}$ if and only if B is a quasi-tilted algebra.
In Section 3 we first introduce the repetitive cluster categories $\mathcal{C}_{F^m}(\mathcal{H})$, which are triangulated categories and are coverings of the corresponding cluster categories $\mathcal{C}(\mathcal{H})$. We then study cluster tilting theory in these triangulated categories. We show that the covering functor $\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ induces a covering functor from the subcategory of projective modules of the endomorphism algebra (called generalized cluster-tilted algebra) of a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ to the cluster-tilted algebra of the corresponding cluster tilting objects in $\mathcal{C}(\mathcal{H})$. ρ_m also induces the corresponding push-down functors between their module categories. The similar result is proved for the covering functor $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$.

2 Basics on cluster-tilted algebras

Let $\mathcal{D}$ be a $k-$linear triangulated category with finite dimensional Hom-spaces over a field k and with Serre duality. We assume that $\mathcal{D}$ is a Krull-Remak-Schmidt category. Let $\mathcal{T}$ be a full subcategory of $\mathcal{D}$ closed under taking direct summands. The quotient category of

$\mathcal{D}$ by $\mathcal{T}$ denoted by $\mathcal{D}/\mathcal{T}$, is by definition, a category with the same objects as $\mathcal{D}$ and the space of morphisms from X to Y is the quotient of group of morphisms from X to Y in $\mathcal{D}$ by the subgroup consisting of morphisms factor through an object in $\mathcal{T}$. The quotient $\mathcal{D}/\mathcal{T}$ is also an additive Krull-Remak-Schmidt category (see for example Lemma 2.1 in [KZ]). For $X, Y \in \mathcal{D}$, we use $Hom(X,Y)$ to denote $Hom_{\mathcal{D}}(X,Y)$ for simplicity, and define that $Ext^k(X,Y) := Hom(X, Y[k])$. For a subcategory $\mathcal{T}$, we say that $Ext^i(\mathcal{T},\mathcal{T}) = 0$ provided that $Ext^i(X,Y) = 0$ for any $X, Y \in \mathcal{T}$. For an object T, add$T$ denotes the full subcategory consisting of direct summands of direct sum of finitely many copies of T. Throughout the article, the composition of morphisms $f : M \to N$ and $g : N \to L$ is denoted by $fg : M \to L$. For basic references on representation theory of algebras and triangulated categories, we refer [Rin1] and [H1].

Fix a triangulated category $\mathcal{D}$, and assume that $\mathcal{T}$ is a functorially finite subcategory of $\mathcal{D}$ (see for example [AS]).

Definition 2.1. *1. $\mathcal{T}$ is called rigid, provided $Ext^1(\mathcal{T},\mathcal{T}) = 0$; in particular, an object T is called rigid provided $Ext^1(T,T) = 0$.*

2. $\mathcal{T}$ is called a cluster tilting subcategory provided $X \in \mathcal{T}$ if and only if $Ext^1(X,\mathcal{T}) = 0$ and $X \in \mathcal{T}$ if and only if $Ext^1(\mathcal{T},X) = 0$. An object T is a cluster tilting object if and only if add$T$ is a cluster tilting subcategory.

Remark 2.2. *1. Not all triangulated categories have cluster tilting subcategories, see for example, the example in Section 5 in [KZ].*

2. In a module category $\Lambda - mod$ of a self-injective algebra Λ, $T \oplus \Lambda$ is a cluster tilting module (=maximal $2-$orthogonal module in [I1, I2]) if and only if T is cluster tilting in $\Lambda - \underline{mod}$.

Remark 2.3. *It was proved in [KZ] that if $\mathcal{T}$ is contravariantly finite and satisfies the condition that $X \in \mathcal{T}$ if and only if $Ext^1(\mathcal{T},X) = 0$, then $\mathcal{T}$ is a cluster tilting subcategory.*

For a triangulated category $\mathcal{D}$ with Serre duality Σ, $\mathcal{D}$ has Auslander-Reiten triangles and $\Sigma = \tau[1]$, where τ is the Auslander-Reiten translation. Denote by $F = \tau^{-1}[1]$.

Lemma 2.4. *Let $\mathcal{D}$ be a triangulated category with Serre duality Σ, and $\mathcal{T}$ a cluster tilting subcategory of $\mathcal{D}$. Then $F\mathcal{T} = \mathcal{T}$.*

Proof. The assertion was proved in [KZ] or [IY]. □

The following results were proved in [KZ], see also [BMRRT][BMR1][KR][IY].

Theorem 2.5. *Let T be a cluster tilting object of a triangulated category $\mathcal{D}$, and $A = End_{\mathcal{D}}T$. Then the following hold:*

1. *(Corollaries 4.4, 4.5 in [KZ]). The functor $Hom(T,-):\mathcal{D}\to A-mod$ induces an equivalence $\mathcal{D}/add(T[1])\cong A-mod$, and A is a Gorenstein algebra of Gorenstein dimension at most* 1.

2. *(Proposition 4.8 in [KZ]). Assume that the field k is algebraically closed. If $B = End_{\mathcal{D}}T'$ is the endomorphism algebra of another cluster tilting object T', then A and B have same representation type.*

Let $T = T_1 \oplus T'$ be a cluster tilting object of a triangulated category $\mathcal{D}$, where T_1 is indecomposable object. Let $T_1^* \to E \xrightarrow{f} T_1 \to T_1^*[1]$ be the triangle with f a minimal right $addT'-$approximation of T_1. It follows from [IY] that $T^* = T_1^* \oplus T'$ is a cluster tilting object and there is a triangle $T_1 \to E' \xrightarrow{g} T_1^* \to T_1[1]$ with g being a minimal right $addT'-$approximation of T_1^*. Let A, B be the endomorphism algebras of cluster tilting objects T, T^* respectively. Denote by S_{T_1}, (or $S_{T_1^*}$) the simple $A-$module corresponding to T_1 (resp. simple $B-$module corresponding to T_1^*). The following proposition is a generalization of Proposition 2.2 in [KR] and Theorem B in [BMR1].

Proposition 2.6. *Let T and T^* be as above. Then $A-mod/addS_{T_1} \approx B-mod/addS_{T_1^*}$.*

Proof. Denote by $G = \mathrm{Hom}(T,-)$. The induced functor $\bar{G}:\mathcal{D}/\mathrm{add}(T[1]) \to A-\mathrm{mod}$ is an equivalence by Theorem 2.5(1). We consider the composition of the functor $\bar{G}$ with the quotient functor $Q: A-\mathrm{mod} \to \frac{\Lambda-\mathrm{mod}}{\mathrm{add}(\mathrm{Hom}(T,T_1^*[1]))}$, which is denoted by G_1. The functor G_1 is full and dense since $\bar{G}$ and Q are. Under the equivalence $\bar{G}$, $T_1^*[1]$ corresponds to $\mathrm{Hom}(T,T_1^*[1])$. For any morphism $f: X\to Y$ in the category $\frac{\mathcal{D}}{\mathrm{add}(T[1])}$, $\bar{G}(f): G(X) \to G(Y)$ factors through $\mathrm{add}(\mathrm{Hom}(T,T_1^*[1]))$ if and only if f factors through $\mathrm{add}T_1^*[1]$. Then G_1 induces an equivalence, denoted by $\bar{G_1}$, from the category $\frac{\mathcal{D}}{\mathrm{add}(T[1]\oplus T_1^*[1]))}$ to the category $\frac{A-\mathrm{mod}}{\mathrm{add}(\mathrm{Hom}(T,\tau T_1^*[1]))}$. Therefore we have that $\frac{A-\mathrm{mod}}{\mathrm{add}(\mathrm{Hom}(T,T_1^*[1]))} \approx \frac{B-\mathrm{mod}}{\mathrm{add}(\mathrm{Hom}(T',T_1[1]))}$. It is easy to prove that $\mathrm{Hom}(T,T_1^*[1])) \cong S_{T_1}$ and $\mathrm{Hom}(T^*,T_1[1]) \cong S_{T_1^*}$ (compare Lemma 4.1 in [BMR1]). Then $A-mod/addS_{T_1} \approx B-mod/addS_{T_1^*}$. The proof is finished. □

From now on, we assume that $\mathcal{H}$ is a hereditary $k-$linear category with finite dimensional Hom-spaces and Ext-spaces. We also assume that $\mathcal{H}$ has tilting objects. The endomorphism algebra of tilting object T in $\mathcal{H}$ is called a quasi-tilted algebra [HRS]. Since $\mathcal{H}$ has tilting objects, $D^b(\mathcal{H})$ has Serre duality [HRS], and has also Auslander-Reiten triangles, the Auslander-Reiten translation is denoted by τ [HRS]. Let $F = \tau^{-1}[1]$ be the automorphism of the bounded derived category $D^b(\mathcal{H})$. We call the orbit category $D^b(\mathcal{H})/<F>$ the cluster category of type $\mathcal{H}$, which is denoted by $\mathcal{C}(\mathcal{H})$ [BMRRT]. For cluster tilting theory in the cluster category $\mathcal{C}(\mathcal{H})$, we refer [BKL][BMRRT][Zh]. The endomorphism algebra $End_{\mathcal{C}(\mathcal{H})}T$ of a cluster tilting object T in $\mathcal{C}(\mathcal{H})$ is called a cluster-tilted algebra of type $\mathcal{H}$. When $\mathcal{H}$ is the module category over a hereditary algebra $H = kQ$, we call the corresponding orbit category the cluster category of H or of Q. In this case the endomorphism algebra of a cluster tilting object is called a cluster-tilted algebra of H [BM], [BMR1], [Zh], [ABS1-3].

Now we give a characterization of cluster-tilted algebras of type $\mathcal{H}$, which generalizes some results in [ABS1], [Zh].

Given any finite-dimensional algebra B, from the $B-$bimodule $\text{Ext}^2(DB, B)$, one can form the trivial extension algebra of B with the bimodule $\text{Ext}^2(DB, B)$: $A = B \ltimes \text{Ext}^2(DB, B)$. It was proved that this trivial extension algebra is a cluster-tilted algebra of H if and only if B is a tilted algebra [ABS1], see also [Zh]. In the following, we generalize the characterization of cluster-tilted algebras to the cluster-tilted algebras of type $\mathcal{H}$. The proof is exactly the same as the proof in [ABS1], we omit it here.

Proposition 2.7. *Let $A = B \ltimes Ext^2(DB, B)$. Then A is a cluster-tilted algebra of type $\mathcal{H}$ for some hereditary abelian category $\mathcal{H}$ if and only if B is a quasi-tilted algebra, i.e. the endomorphism algebra of a tilting object in $\mathcal{H}$.*

3 Intermediate covers of cluster tilted algebras of type $\mathcal{H}$

As in the previous section, $\mathcal{H}$ denotes a hereditary $k-$linear category with finite dimensional Hom-spaces and Ext-spaces. We assume that $\mathcal{H}$ has tilting objects. Since $\mathcal{H}$ has tilting objects, $D^b(\mathcal{H})$ has Serre duality, and also Auslander-Reiten translate τ (AR-translate for short)[HRS]. Let $F = \tau^{-1}[1]$ be the automorphism of the bounded derived category $D^b(\mathcal{H})$. Fix a positive integer m throughout this section.
We consider the orbit category $D^b(\mathcal{H})/ < F^m >$, which is by definition a $k-$linear category whose objects are the same in $D^b(\mathcal{H})$, and whose morphisms are given by:

$$\text{Hom}_{D^b(\mathcal{H})/<F^m>}(\widetilde{X}, \widetilde{Y}) = \oplus_{i\in\mathbf{Z}}\text{Hom}_{D^b(\mathcal{H})}(X, (F^m)^i Y).$$

Here X and Y are objects in $D^b(\mathcal{H})$, and $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding objects in $D^b(\mathcal{H})/ < F^m >$ (although we shall sometimes write such objects simply as X and Y).

Definition 3.1. *The orbit category $D^b(\mathcal{H})/ < F^m >$ is called the repetitive cluster category of type $\mathcal{H}$. We denote it by $\mathcal{C}_{F^m}(\mathcal{H})$.*

Remark 3.2. *When $m = 1$, we get back to the usual cluster category $\mathcal{C}(\mathcal{H})$, which was introduced by Buan-Marsh-Reineke-Reiten-Todorov in [BMRRT], and also by Caldero-Chapoton-Schiffler in [CCS] for A_n case.*

The repetitive cluster categories $\mathcal{C}_{F^m}(\mathcal{H})$ serve as intermediate categories between the cluster categories $\mathcal{C}(\mathcal{H})$ and derived categories $D^b(\mathcal{H})$. Similarly as for the case of cluster categories, for any positive integer m, we have a natural projection functor $\pi_m : D^b(\mathcal{H}) \rightarrow \mathcal{C}_{F^m}(\mathcal{H})$. If $m = 1$, the projection functor π_m is simply denoted by π.
Now we define a functor $\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \longrightarrow \mathcal{C}(\mathcal{H})$, which sends objects $\tilde{X}$ in $\mathcal{C}_{F^m}(\mathcal{H})$ to objects $\tilde{X}$ in $\mathcal{C}(\mathcal{H})$ and morphisms $f : \tilde{X} \rightarrow \tilde{Y}$ in $\mathcal{C}_{F^m}(\mathcal{H})$ to the morphisms $f : \tilde{X} \rightarrow \tilde{Y}$ in $\mathcal{C}(\mathcal{H})$.
It is easy to check that $\pi = \rho_m \circ \pi_m$.
One can identify the set ind$\mathcal{C}(\mathcal{H})$ with the fundamental domain for the action of F on ind$D^b(\mathcal{H})$ [BMRRT]. Passing to the orbit category $\mathcal{C}_{F^m}(\mathcal{H})$, one can view ind$\mathcal{C}(\mathcal{H})$ as a (usually not full) subcategory of ind$\mathcal{C}_{F^m}(\mathcal{H})$.

Proposition 3.3. *1. $\mathcal{C}_{F^m}(\mathcal{H})$ is a triangulated category with Auslander-Reiten triangles and Serre functor $\Sigma = \tau[1]$, where τ is the AR-translate in $\mathcal{C}_{F^m}(\mathcal{H})$, which is induced from AR-translate in $D^b(\mathcal{H})$.*

2. *The projections $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$ and $\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \longrightarrow \mathcal{C}(\mathcal{H})$ are triangle functors and also covering functors.*

3. *$\mathcal{C}_{F^m}(\mathcal{H})$ is a fractional Calabi-Yau category of CY-dimension $\frac{2m}{m}$.*

4. *$\mathcal{C}_{F^m}(\mathcal{H})$ is a Krull-Remak-Schmidt category.*

5. *$\mathrm{ind}\mathcal{C}_{F^m}(\mathcal{H}) = \bigcup_{i=0}^{i=m-1}(\mathrm{ind}F^i(\mathcal{C}(\mathcal{H})))$.*

Proof. 1. It follows from [K1] that $\mathcal{C}_{F^m}(\mathcal{H})$ is a triangulated category. The remaining claims follow from Proposition 1.3 [BMRRT].

2. It is proved in Corollary 1 in Section 8.4 of [K1] that $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$ is a triangle functor. It is easy to check that $\pi \circ F^m \cong \pi$. By the universal property of the orbit category $D^b(\mathcal{H})/ < F^m >$ [K1], we obtain a triangle functor $\rho : \mathcal{C}_{F^m}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ satisfying that $\rho\pi_m = \pi$, which turns out to be the functor ρ_m.

3. The Serre functor $\Sigma = \tau[1]$ in $\mathcal{C}_{F^m}(\mathcal{H})$ satisfies that $\Sigma^m = \tau^m[m] = F^m[2m] \cong [2m]$. Therefore $\mathcal{C}_{F^m}(\mathcal{H})$ is a fractional Calabi-Yau category with CY-dimension $\frac{2m}{m}$.

4. The proof given in Proposition 1.6 [BMRRT] for $m = 1$, can be modified to work for any positive value of m.

□

We note that if the hereditary abelian category $\mathcal{H}$ is equivalent to the module category of a finite dimensional hereditary algebra H, then the indecomposable objects in $\mathcal{C}(\mathcal{H})$ are of form $\tilde{M}$ or of form $\tilde{P[1]}$, where M is an indecomposable $H-$module and $P[1]$ is the first shift of an indecomposable projective $H-$module P. If the hereditary abelian category $\mathcal{H}$ is not equivalent to the module category of a finite dimensional hereditary algebra H, then the indecomposable objects in $\mathcal{C}(\mathcal{H})$ are of form $\tilde{M}$, where M is an indecomposable object in $\mathcal{H}$.
Now we discuss the cluster tilting objects in $\mathcal{C}_{F^m}(\mathcal{H})$. Denoted by $F = \tau^{-}[1]$, which can be viewed an automorphism of $D^b(\mathcal{H})$ or of $\mathcal{C}_{F^m}(\mathcal{H})$. The following proposition is a generalization of Lemma 4.14 in [KZ].

Proposition 3.4. *An object T in $\mathcal{C}_{F^m}(\mathcal{H})$ is a cluster tilting object if and only if $\pi_m^{-1}(addT)$ is a cluster tilting subcategory of $D^b(\mathcal{H})$*

Proof. We only give a detailed proof in the case $\mathcal{H}$ is equivalent to the module category of a finite dimensional hereditary algebra H. The proof in case $\mathcal{H}$ is not of the form is similar.
Suppose that $\mathcal{H} \approx H - mod$, where H is a finite dimensional hereditary algebra over a field k. For an object T in $\mathcal{C}_{F^m}(\mathcal{H})$, we denote $\mathcal{T} = \pi_m^{-1}(addT)$, which is a full subcategory of $D^b(H)$. It is easy to prove that $F(\mathcal{T}) = \mathcal{T}$ in $D^b(\mathcal{H})$ if and only if $F(addT) = addT$ in $\mathcal{C}_{F^m}(\mathcal{H})$.

Suppose $\mathcal{T}$ is a cluster tilting subcategory of $D^b(H)$. Then $F\mathcal{T} = \mathcal{T}$ by Lemma 2.4 or Proposition 4.7 [KZ]. Hence $F(addT) = addT$ in $\mathcal{C}_{F^m}(\mathcal{H})$. We denote by $\mathcal{T}'$ the intersection of $\mathcal{T}$ with the additive subcategory $\mathcal{C}'$ generated by all $H-$modules as stalk complexes of degree 0 together with $H[1]$. Then we have that $\mathcal{T} = \{F^n(\mathcal{T}')|n \in \mathbf{Z}\}$. Now $\pi_m(\mathcal{T}) = \pi_m(\bigcup_{i=0}^{i=m-1} F^i(\mathcal{T}'))$, denoted by $\mathcal{T}_1$. For any pair of objects $\tilde{T_1}, \tilde{T_2} \in \mathcal{T}_1$, there are $T_1, T_2 \in \mathcal{T}'$ such that $\tilde{T_1} = F^t(\pi_m(T_1)), \tilde{T_2} = F^s(\pi_m(T_2))$ with $0 \leq t, s \leq m-1$. Then $Ext^1(\tilde{T_1}, \tilde{T_2}) = Hom(\tilde{T_1}, \tilde{T_2}[1]) \cong \oplus_{n\in Z} Hom_{D^b(H)}(F^s(T_1), (F^m)^n F^t(T_2[1])) = \oplus_{n\in Z} Hom_{D^b(H)}(T_1, F^{mn+t-s}T_2[1])$. By an easy computation, one has that $Hom_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = 0$ if $nm+t-s \leq -2$ or $nm+t-s \geq 1$. When $nm+t-s = -1$, $Hom_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = Hom_{D^b(H)}(T_1, F^{-1}T_2[1]) = Hom_{D^b(H)}(T_1, \tau T_2) \cong DExt_{D^b(H)}(T_2, T_1)$, which equals 0 by the fact that $\mathcal{T}$ is a cluster tilting subcategory of $D^b(H)$. When $nm+t-s = 0$, $Hom_{D^b(H)}(T_1, F^{mn+t-s}T_2[1]) = Hom_{D^b(H)}(T_1, T_2[1]) = Ext_{D^b(H)}(T_1, T_2)$, which equals 0 by the fact that $\mathcal{T}$ is a cluster tilting subcategory of $D^b(H)$. Therefore $Ext^1(\tilde{T_1}, \tilde{T_2}) = 0$, i.e. $\mathcal{T}_1$ is rigid in $\mathcal{C}_{F^m}(\mathcal{H})$.
If there are indecomposable objects $\tilde{X} = \pi_m(X) \in \mathcal{C}_{F^m}(H)$ with $X \in D^b(\mathcal{H})$ satisfying $Ext^1(\mathcal{T}_1, \tilde{X}) = 0$, then $Ext^1(F^n\mathcal{T}', X) = 0$ for any n, and then $Ext^1(\mathcal{T}, X) = 0$. Hence X is in $\mathcal{T}$ since $\mathcal{T}$ is a cluster tilting subcategory. Thus $\tilde{X} \in \mathcal{T}_1$. This proves that the image $\mathcal{T}_1$ of $\mathcal{T}$ under π_m is a cluster tilting subcategory of $\mathcal{C}_{F^m}(H)$.
Conversely, from $\mathcal{T} = \pi_m^{-1}(\mathcal{T}_1)$ and $F(\mathcal{T}_1) = \mathcal{T}_1$, we get $F(\mathcal{T}) = \mathcal{T}$. As above we denote by $\mathcal{T}'$ the intersection of $\mathcal{T}$ with the additive subcategory $\mathcal{C}'$ generated by all $H-$modules as stalk complexes of degree 0 together with $H[1]$. Then $\mathcal{T} = \{F^n(\mathcal{T}')|n \in \mathbf{Z}\}$ and $\mathcal{T}_1 = \pi_m(\mathcal{T}) = \pi_m(\bigcup_{i=o}^{i=m-1} F^i(\mathcal{T}'))$. From $\mathcal{T}_1$ being contravariantly finite, we have $\mathcal{T}$ is also contravariantly finite. Since $Ext^1(\mathcal{T}_1, \mathcal{T}_1) \cong \oplus_{n\in Z} Ext^1_{D^b(H)}(\bigcup_{i=o}^{i=m-1} F^i(\mathcal{T}'), F^n(\bigcup_{i=o}^{i=m-1} F^i(\mathcal{T}')) = 0$, we have that $Ext^1_{D^b(H)}(F^m\mathcal{T}', F^n\mathcal{T}') \cong Ext^1_{D^b(H)}(\mathcal{T}', F^{n-m}\mathcal{T}') = 0$. This proves that $\mathcal{T}$ is a rigid subcategory. Now if $X \in D^b(H)$ satisfies $Ext^1_{D^b(H)}(\mathcal{T}, X) = 0$, then $Ext^1_{\mathcal{C}_{F^m}(\mathcal{H})}(F^i(\mathcal{T}_1), \tilde{X}) = 0$, $\forall 0 \leq i \leq m-1$. It follows that $\tilde{X} \in \mathcal{T}_1$, hence $X \in \mathcal{T}$. Similarly, if $X \in D^b(H)$ satisfies $Ext^1_{D^b(H)}(X, \mathcal{T}) = 0$, then $X \in \mathcal{T}$. □

From Proposition 3.4 above and Lemma 4.14 in [KZ], we have a one-to-one correspondence between the three sets: the set of cluster tilting subcategories in $D^b(\mathcal{H})$; the set of cluster tilting subcategories in $\mathcal{C}_{F^m}(\mathcal{H})$; the set of cluster tilting subcategories in $\mathcal{C}(\mathcal{H})$, via triangle covering functors: $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$ and $\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \longrightarrow \mathcal{C}(\mathcal{H})$.

Theorem 3.5. *Let $\mathcal{H}$ be a hereditary abelian category with tilting objects. Let $T \in \mathcal{C}(\mathcal{H})$.*

1. *T is a cluster tilting object in cluster category $\mathcal{C}(\mathcal{H})$ if and only if $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ if and only if $\pi^{-1}(addT)$ is a cluster tilting subcategory in $D^b(\mathcal{H})$.*

2. *For any tilting object T' in $\mathcal{H}$, $\oplus_{i=0}^{i=m-1} F^iT'$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$, and any cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ arises in this way, i.e. there is a hereditary abelian category $\mathcal{H}'$, which is derived equivalent to $\mathcal{H}$, and a tilting object T in $\mathcal{H}'$ such that the cluster tilting object is induced from T.*

Proof. 1. It follows Lemma 4.14 in [KZ] or the special case of Proposition 3.4 where $m = 1$, that T is a cluster tilting object in $\mathcal{C}(\mathcal{H})$ if and only if $\pi^{-1}(addT)$ is a cluster tilting subcategory in $D^b(\mathcal{H})$. By Proposition 3.4, we have that $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ if and only if $\pi_m^{-1}(add(\rho_m^{-1}(T)))$ is a cluster tilting subcategory in $D^b(\mathcal{H})$. Since $\pi = \rho_m\pi_m$, $\pi(\pi_m^{-1}(add(\rho_m^{-1}(T))) = T$, we have that $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ if and only if T is a cluster tilting object in $\mathcal{C}(\mathcal{H})$.

2. For any tilting object T' in $\mathcal{H}$, from [BMRRT] and [Zh], T' is a cluster tilting object in $\mathcal{C}(\mathcal{H})$. Hence $\oplus_{i=0}^{i=m-1}F^iT'$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$ by the first part of the theorem. Suppose M is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$. Then by the first part of the theorem, $\rho_m(M)$ is a cluster tilting object in the cluster category $\mathcal{C}(\mathcal{H})$. Therefore $\rho_m(M)$ is induced from a tilting object of a hereditary abelian category $\mathcal{H}'$, which is derived equivalent to $\mathcal{H}$ [Zh, BMRRT]. Then M is induced from a tilting object of $\mathcal{H}'$.

□

Definition 3.6. *We call the endomorphism algebras* $End_{\mathcal{C}_{F^m}(\mathcal{H})}T$ *of cluster tilting objects* T *in the repetitive cluster category* $\mathcal{C}_{F^m}(\mathcal{H})$ *the generalized cluster-tilted algebras of type* $\mathcal{H}$*, or simply the generalized cluster-tilted algebras.*

Now we study the representation theory of generalized cluster-tilted algebras. We recall that $\pi_m : D^b(\mathcal{H}) \longrightarrow \mathcal{C}_{F^m}(\mathcal{H})$ is the projection.

Theorem 3.7. *Let* T *be a tilting object in* $\mathcal{H}$, $\widetilde{A} = End_{\mathcal{C}_{F^m}(\mathcal{H})}(\oplus_{i=0}^{i=m-1}F^iT)$ *the generalized cluster-tilted algebra.*

1. $\widetilde{A}$ *has a Galois covering* $\pi_m : \pi^{-1}(addT) \to \rho_m^{-1}(addT)$ *which is the restriction of the projection* $\pi_m : D^b(\mathcal{H}) \to \mathcal{C}_{F^m}(\mathcal{H})$.

2. *The projection* π_m *induces a push-down functor* $\tilde{\pi}_m : \frac{D^b(\mathcal{H})}{add\{\tau^nT[-n] \mid n\in Z\}} \longrightarrow \tilde{A}-mod$.

3. *If* T' *is a tilting object in* $\mathcal{H}$*, then the generalized cluster tilted algebra* $\widetilde{A}' = End_{\mathcal{C}_{F^m}(\mathcal{H})}(\oplus_{i=0}^{i=m-1}F^iT')$ *has the same representation type as* A.

Proof. (1). Let $\mathcal{T} = add(\{\ F^i(T) \mid i \in \mathbf{Z}\ \})$. $\mathcal{T} = \pi^{-1}(addT)$ is a cluster tilting subcategory of $D^b(\mathcal{H})$. Hence by Proposition 3.4 , $\pi_m(\mathcal{T}) = \rho_m^{-1}(addT)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{H})$. By Theorem 2.5, we have the equivalence $Hom_{\mathcal{C}_{F^m}(\mathcal{H})}(\oplus_{i=0}^{i=m-1}\pi_m(F^i(T)), -) : \frac{\mathcal{C}_{F^m}(\mathcal{H})}{add(\oplus_{i=0}^{i=m-1}\pi_m(F^i(T)))} \to \tilde{A} - mod$. Under this equivalence, the subcategory $add(\pi_m(\mathcal{T}))$ correspondences to the subcategory of projective $\tilde{A}-$modules.
The projection π_m sends $\mathcal{T}$ to $\pi_m(\mathcal{T})$. Thus $\pi_m|_{\mathcal{T}} : \mathcal{T} \longrightarrow \rho_m^{-1}(\mathcal{T})$ is a Galois covering with Galois group generated by F^m.
(2). By Theorem 3.3 and Corollary 4.4 in [KZ] there are equivalences $D^b(H)/\mathcal{T}[1] \cong mod(\mathcal{T})$ and $\mathcal{C}_m(\mathcal{H})/(\pi_m(\mathcal{T}[1])) \cong mod(\pi_m(\mathcal{T}))$. We define the induced functor $\bar{\pi}_m$ as follows: $\bar{\pi}_m(X) := \pi_m(X)$ for any object $X \in D^b(H)/\mathcal{T}[1]$, and $\bar{\pi}_m(\underline{f}) := \underline{\pi_m(f)}$ for any morphism $\underline{f} : X \to Y$ in $D^b(H)/\mathcal{T}$. Clearly $\bar{\pi}_m$ is well-defined and makes the following diagram commutative:

$$\begin{array}{ccc} D^b(\mathcal{H}) & \xrightarrow{\pi_m} & \mathcal{C}_m(\mathcal{H}) \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ D^b(\mathcal{H})/\mathcal{T}[1] & \xrightarrow{\bar{\pi}_m} & \mathcal{C}_m(\mathcal{H})/\pi(\mathcal{T})[1]. \end{array}$$

Where P_1, P_2 are the natural quotient functors. Then $\bar{\pi}_m$ is a covering functor from $D^b(\mathcal{H})/\mathcal{T}[1]$ to $\mathcal{C}_{F^m}/\pi(\mathcal{T}[1])$, i.e., it is a covering functor from $D^b(\mathcal{H})/\mathcal{T}[1]$ to $\tilde{A}-mod$ ($\approx mod(\pi_m(\mathcal{T}))$.
(3). This is a direct consequence of Theorem 2.2 □

Similarly as above, the triangle covering functor $\rho_m : \mathcal{C}_m(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ induces a covering functor from $\tilde{A}$ to the cluster-tilted algebra $End_{\mathcal{C}(\mathcal{H})}T$ indicated as the following Theorem.

Theorem 3.8. *Let T be a tilting object in $\mathcal{H}$, $A = End_{\mathcal{C}(\mathcal{H})}T$ and $\widetilde{A} = End_{\mathcal{C}_{F^m}(\mathcal{H})}(\oplus_{i=0}^{i=m-1}F^iT)$ the generalized cluster-tilted algebra.*

1. *$\rho_m : \mathcal{C}_{F^m}(\mathcal{H}) \to \mathcal{C}(H)$ restricted to the cluster tilting subcategory $add(\bigcup_{i=0}^{i=m-1} F^iT)$ induces a Galois covering of A.*
2. *The functor ρ_m also induces a push-down functor $\tilde{\rho_m} : \tilde{A}-mod \longrightarrow A-mod$.*

Proof. The strategy of the proof is almost the same as that of Theorem 3.7, we present it here for the convenience of reader.
(1). By Theorem 3.5, $\rho_m^{-1}(T)$ is a cluster tilting object in $\mathcal{C}_{F^m}(\mathcal{C})$, and $\rho_m^{-1}(T) = \oplus_{i=1}^{i=m-1}F^i(T)$. By Theorem 2.5, we have the equivalence $Hom_{\mathcal{C}_{F^m}(\mathcal{H})}(\rho_m^{-1}(T), -) : \frac{\mathcal{C}_{F^m}(\mathcal{H})}{add(\rho_m^{-1}(T))} \to \tilde{A}-mod$. Under this equivalence, the subcategory $add(\rho_m^{-1}(T))$ corresponds to the subcategory of projective $\tilde{A}-$modules.
The triangle functor ρ_m sends add$\rho_m^{-1}(T)$ to add$T$. Thus $\rho_m|_{add\rho_m^{-1}(T)} : add\rho_m^{-1}(T) \longrightarrow addT$ is a Galois covering with Galois group Z_m.
(2). By Theorem 3.3 and Corollary 4.4 in [KZ], there is an equivalence $\mathcal{C}_m(\mathcal{H})/(add\rho_m^{-1}(T)[1]) \cong \tilde{A}-mod$. We define the induced functor $\bar{\rho}_m$ as follows: $\bar{\rho}_m(X) := \rho_m(X)$ for any object $X \in \mathcal{C}_m(\mathcal{H})/(\rho_m^{-1}(T))[1]$, and $\bar{\rho}_m(\underline{f}) := \underline{\rho_m(f)}$ for any morphism $\underline{f} : X \to Y$ in $\mathcal{C}_m(\mathcal{H})/(\rho_m^{-1}(T))$. Clearly $\bar{\rho}_m$ is well-defined and makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{C}_{F^m}(\mathcal{H}) & \xrightarrow{\rho_m} & \mathcal{C}(\mathcal{H}) \\ {\scriptstyle P_1'}\downarrow & & \downarrow{\scriptstyle P_2'} \\ \mathcal{C}_{F^m}(\mathcal{H})/add(\rho_m^{-1}(T))[1]) & \xrightarrow{\bar{\rho}_m} & \mathcal{C}(\mathcal{H})/add(T[1]). \end{array}$$

Where P_1', P_2' are the natural quotient functors. Then $\bar{\rho}_m$ is a covering functor from $\mathcal{C}_m(\mathcal{H})/(add\rho_m^{-1}(T)[1])$ to $\mathcal{C}/add(T[1])$, i.e, it is a covering functor from $\tilde{A}-mod$ to $A-mod$. □

Remark 3.9. *By Proposition 2.7, see also [ABS,Zh], the cluster-tilted algebra A of type $\mathcal{H}$ can be written as a trivial extension $A = B \ltimes M$, where $M = Ext_B^2(DB, B)$. Then*

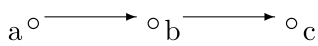
A has as $\mathbf{Z}-$*covering the following (infinite dimensional) matrix algebra (i.e. the cluster repetitive algebra in [ABS3]):*

$$A_\infty = \begin{bmatrix} \ddots & & & & \\ \ddots & B & & & \\ & M & B & & \\ & & M & B & \\ & & & \ddots & \ddots \end{bmatrix}$$

On the other hand, $A = B \ltimes M$ *is also a* Z_m-*graded algebra. Then* A *has a* Z_m-*covering* $A\sharp Z_m$*, the smash product of graded algebra* A *with group* Z_m*[CM].*

Examples

1. Let $D^b(H)$ be the (bounded) derived category of hereditary algebra H, where H is the path algebra of the quiver :

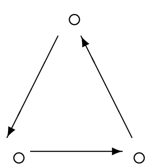


If we take $\mathcal{T}$ to be the subcategory generated by $\{\tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n \in Z\}$, then $\mathcal{T}$ is a cluster tilting subcategory of $D^b(H)$ and $D^b(H)/\mathcal{T} \cong A_\infty - mod$ where A_∞ is the algebra with quiver

$$A_\infty^\infty : \cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \cdots$$

and with $rad^2 = 0$ [KZ].

2. Let $m = 1$. We consider the cluster category $\mathcal{C}(H)$. If we take $T = P_a \oplus P_c \oplus S_a$, then T is a cluster tilting object of $\mathcal{C}(H)$ and $\mathcal{C}(H)/(addT) \cong A - mod$ where A is the algebra with quiver



and with $rad^2 = 0$.

3. Let $m = 2$. We consider the repetitive cluster category $\mathcal{C}_{F^2}(A)$. If we take $\mathcal{T}'$ to be the subcategory generated by $\{\tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n = 0, 1\}$, then $\mathcal{T}'$ is a cluster tilting subcategory of $\mathcal{C}_{F^2}(A)$ and $\mathcal{C}_{F^2}(A)/\mathcal{T}' \cong A_1 - mod$ where A_1 is the algebra with quiver

$$
\begin{array}{cccccc}
 & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\
Q_1 : & \uparrow & & & & \downarrow \\
 & \circ & \longleftarrow & \circ & \longleftarrow & \circ
\end{array}
$$

and with $rad^2 = 0$.

4. Let $m = 3$. We consider the repetitive cluster category $\mathcal{C}_{F^3}(A)$. If we take $\mathcal{T}''$ to be the subcategory generated by $\{\tau^{-n}P_a[n], \tau^{-n}S_a[n], \tau^{-n}P_c[n] \mid n = 0, 1, 2\}$, then $\mathcal{T}''$ is a cluster tilting subcategory of $\mathcal{C}_{F^3}(A)$ and $\mathcal{C}_{F^3}(A)/\mathcal{T}'' \cong A_2 - mod$ where A_2 is the algebra with quiver

$$
\begin{array}{cccccccccc}
 & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\
Q_2 : & \uparrow & & & & & & & \swarrow & \\
 & \circ & \longleftarrow & \circ & \longleftarrow & \circ & \longleftarrow & \circ & &
\end{array}
$$

and with $rad^2 = 0$.

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