A Criterion for Coils

Bin Zhu*

Department of Mathematical Sciences, Tsinghua
University, 100084 Beijing, P.R.China
E-mail: bzhu@math.tsinghua.edu.cn

Abstract. We give an axiomatic description of coils, which improves the Assem-Skowroński characterization of coils by removing two of the conditions in Theorem 4.2 in [4]. This gives a suitable generalization of the D'Este-Ringel characterization of coherent tubes given in [7].

Keywords: translation quiver, coil, length functions.

Mathematics Subject Classification. 05E10, 16G10, 16G20.

1. Introduction

One of the main concerns of the representation theory of a finite dimensional algebra A is the components of the Auslander-Reiten quiver Γ_A : the vertices of Γ_A are the isomorphism classes $[X]$ of the indecomposable $A-$modules X, $[X] \rightarrow [Y]$ is an arrow in Γ_A if there is an irreducible map from $[X]$ to $[Y]$, the translation in Γ_A is the Auslander-Reiten translation $\tau_A = \mathrm{DTr}A$. If A is connected and not representation-finite, all the components of Γ_A are countable. The present paper concerns a special class of such components which arise rather frequently for tame algebras: the coils introduced by Assem and Skowroński in [3, 4].

In their study of tame algebras of polynomial growth, Assem and Skowroński in [3, 4] introduced the notions of admissible operations on a component of Γ_A, which generalized those of ray insertions or coray insertions [7, 9]. These admissible operations are named as $ad1)$, $ad2)$, $ad3)$, $ad1)^*$, $ad2)^*$, $ad3)^*$. These definitions can be given on a translation quiver, for details, we refer to [3]. Using these operations, they defined and described components of Auslander-Reiten quiver, called coils. A translation quiver Γ without multiple arrows is called a coil if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \cdots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $0 \leq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation. Thus, any stable tube is a coil. A coherent

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tube [7] is a coil having the property that each admissible operation in the sequence defining it is of the form $ad1)$ or $ad1)^*$. A quasi-tube is a coil having the property that each admissible operation in the sequence defining it is of the form $ad1)$, $ad1)^*$, $ad2)$ or $ad2)^*$. Coils play an important role in the study of tame algebras [2, 6, 8, 10, 11, 12], in particular, in the study of algebras of polynomial growth, which received recently more attentions, see [5, 8, 12] and their references.

Our aim in the paper is to give an axiomatic description of coils, which improves the Assem-Skowroński characterization of coils: we show that two of their conditions just can be dropped. This gives a suitable generalization of the D'Este-Ringel characterization of coherent tubes given in [7]. Our proof is based on a careful analysis of length functions on such components.

Before we state our main result, we need the notion of a crowned cylinder (for the definition, see [4]). It is known that the underlying topological space of the full translation subquiver of a coil consisting of all points except those which are projective-injective middle terms of a mesh with three middle terms is homeomorphic to a crowned cylinder, for details we refer to section 4.2 in [4]. As in 4.2 in [4], let Γ be a translation quiver without multiple arrows containing a cyclical path, and let Γ' denote the full translation subquiver of Γ consisting of all vertices except those which are projective-injective middle terms of a mesh with three middle terms. Assume the underlying topological space $|\Gamma'|$ of Γ' is homeomorphic to a crowned cylinder, then Γ contains a (maximal) tube as a cofinite full translation subquiver. And then we may consider rays and corays, and denote the ray starting at x by $[x, \infty[$, the coray ending at y by $]\infty, y]$. Any mesh in Γ has at most three middle terms. A mesh with exactly three middle terms will be called exceptional, and a projective middle term of an exceptional mesh will be called exceptional projective. Other meshes and projectives will be called ordinary.

Now we state the main theorem, which improves Theorem 4.2 in [4] by removing the conditions (c2) and (c3) there.

Theorem. *Let Γ be a translation quiver without multiple arrows and containing a cyclical path. Then Γ is a coil if and only if it satisfies the following conditions:*

(c1) Let Γ' denote the full translation subquiver of Γ consisting of all vertices except those which are projective-injective middle terms of a mesh with three middle terms. Then the underlying topological space of Γ' is homeomorphic to a crowned cylinder.

(c2) For any projective vertex $p \in \Gamma_0$, or injective vertex $q \in \Gamma_0$, the ray $[p, \infty[$, or the coray $]\infty, q]$, respectively, exists, and if p is exceptional projective, or q is exceptional injective, then the ray $[p, \infty[$, or the coray $]\infty, q]$, does not contain any projective vertex, or injective vertex, respectively.

(c3) The $\tau-$orbit of any projective vertex or injective vertex contains a vertex which belongs to a cyclical path.

(c4) There exists a length function f on Γ.

2. Preliminary definitions and basic lemmas

Let $\mathbb{N}$ be the set of natural numbers.

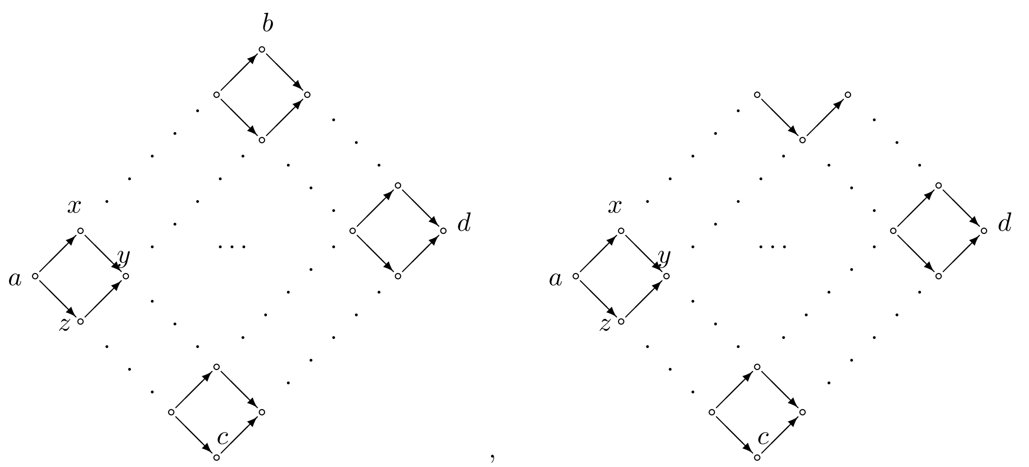
Definition *Let Γ be a translation quiver. A function $f : \Gamma_0 \longrightarrow \mathbb{N}$ is called a length function for Γ if f satisfies the following properties:*

(1) (Additivity). For any non-projective vertex x of Γ, one has $f(x)+f(\tau x) = \Sigma_{y\to x} f(y)$.

(2). If z is a projective vertex of Γ, then $f(z) = 1 + \Sigma_{y\to z} f(y)$.

(2*). *If x is an injective vertex of Γ, then $f(x) = 1 + \Sigma_{x\to y} f(y)$.*

In the following, we always assume that Γ is a translation quiver without multiple arrows, with a length function f. We denote respectively by $\diamondsuit_{abcd}$, $\diamondsuit_{a-cd}$ the following full translation subquivers of Γ (compare [1]),



A mesh is just a minimal rectangle. For a mesh $\diamondsuit_{axzy}$, we denote by $\delta_{\diamondsuit_{axzy}}$ (sometimes denoted by δ_y, for simplicity) $f(a) + f(y) - f(x) - f(z)$, which is called the defect of mesh $\diamondsuit_{axzy}$. If $\delta_y = 0$, the mesh is called a complete mesh, otherwise, it is called an incomplete mesh. We denote by the interval $[a,\ b]$ the unique sectional path from a to b if it exists, and denote by $\delta_{\diamondsuit_{abcd}}$ the sum of defects of all meshes contained in rectangle $\diamondsuit_{abcd}$(see the picture above) i.e.,

$$\delta_{\diamondsuit_{abcd}} = \sum_{e\in\diamondsuit_{abcd}\setminus\{[a,\ b]\cup[a,\ c]\}} \delta_e.$$

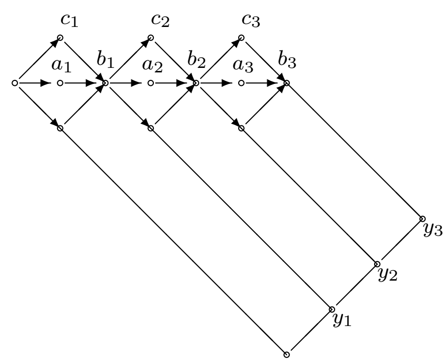
The following three lemmas are easy to prove, but they will play an important role in the proof of the Theorem.

Lemma 2.1. *Let* Γ *be a translation quiver with a length function* f, $\diamondsuit_{abcd}$ *a full translation subquiver of* Γ *(the shape is as above). Then* $f(a) + f(d) = f(b) + f(c) + \delta_{\diamondsuit_{abcd}}$.

Proof. It is well-known, see Lemma 2.1 in [1].

The following lemma is similar to Lemma 4.4 in [4].

Lemma 2.2. *Let* Γ *be a translation quiver with a length function* f. *We consider the full translation subquiver of* Γ *of the form.*

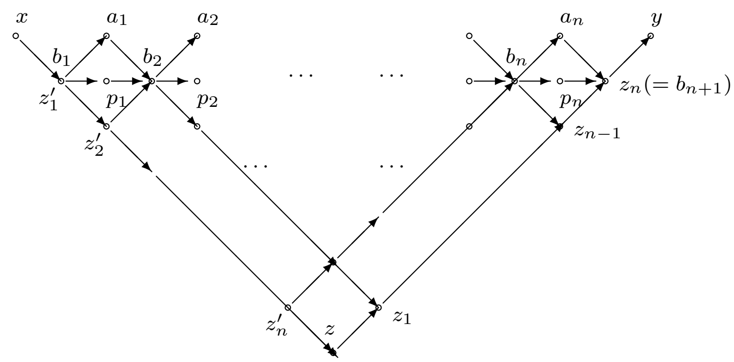


$$\begin{aligned} Then \quad f(c_2) - f(y_2) &= f(a_3) - f(y_3) + \delta_{\diamondsuit_{c_2-y_2y_3}}, \\ f(a_1) - f(y_1) &= f(c_2) - f(y_2) + \delta_{\diamondsuit_{b_1c_2y_1y_2}} + \delta_{a_2} \\ &= f(a_3) - f(y_3) + \delta_{\diamondsuit_{c_2-y_2y_3}} + \delta_{\diamondsuit_{b_1c_2y_1y_2}} + \delta_{a_2}. \end{aligned}$$

Proof. By applying Lemma 2.1 to the rectangle $\diamondsuit_{c_2-y_2y_3}$, one obtains that $f(c_2) + f(y_3) = f(y_2) + f(a_3) + \delta_{\diamondsuit_{c_2-y_2y_3}}$. It follows that $f(c_2) - f(y_2) = f(a_3) - f(y_3) + \delta_{\diamondsuit_{c_2-y_2y_3}}$. By applying Lemma 2.1 again, we get $f(b_1) + f(y_2) = f(y_1) + f(a_2) + f(c_2) + \delta_{\diamondsuit_{b_1c_2y_1y_2}}$, which follows that $f(b_1) + \delta_{a_2} - f(a_2) - f(y_1) = f(c_2) - f(y_2) + \delta_{\diamondsuit_{b_1c_2y_1y_2}} + \delta_{a_2}$. This, combined with the equality $f(b_1) + \delta_{a_2} = f(a_1) + f(a_2)$, follows that

$$\begin{aligned} f(a_1) - f(y_1) &= f(c_2) - f(y_2) + \delta_{\diamondsuit_{b_1c_2y_1y_2}} + \delta_{a_2} \\ &= f(a_3) - f(y_3) + \delta_{\diamondsuit_{c_2-y_2y_3}} + \delta_{\diamondsuit_{b_1c_2y_1y_2}} + \delta_{a_2}. \end{aligned}$$

Lemma 2.3. *Suppose* Γ *contains a full translation subquiver of the form*



where p_1 is projective, and a_n or p_n is injective. Then if n is odd, p_n must be injective, if n is even, then a_n must be injective, and in each case, $f(x) - f(z) \geq 1,\ f(y) - f(z) \geq 1$.

Proof. The proof of the first part is similar to Corollary 4.4 in [4], but for the completeness, we present it here. Assume a_n is injective in case n being odd, then by Lemma 2.2 and $f(a_n) - f(z_n) \geq 1$, we have $f(a_1) - f(z_1) \geq 1$. But by applying Lemma 2.1 for the rectangle $\diamondsuit_{z_1' a_1 z z_1}$, we obtain that $f(z) + f(p_1) + f(a_1) + \delta_{\diamondsuit_{z_1' a_1 z z_1}} = f(z_1') + f(z_1)$. Therefore, $f(z) + 2 + \delta_{\diamondsuit_{z_1' a_1 z z_1}} \leq 0$. It is a contradiction. The proof for even case is similar.

We prove now the second part. By Lemma 2.2, we have $f(x) - f(z) \geq f(p_1) - f(z_1),\quad f(p_1) - f(z_1) \geq f(a_2) - f(z_2)$, and so on. Then

$$f(x) - f(z) \ \geq f(p_1) - f(z_1) \ \geq \cdots \ \geq \lambda_n,$$

where $\lambda_n = f(p_n) - f(z_n)$ if n is odd, and $\lambda_n = f(a_n) - f(z_n)$ if n is even. Then $f(x) - f(z) \geq 1$.

Similarly, if n is odd, we have $f(y) - f(z) \ \geq f(p_n) - f(z_n') \ \geq \cdots \ \geq f(p_1) - f(z_1') \ \geq 1$, and if n is even, $f(y) - f(z) \ \geq f(a_n) - f(z_n') \ \geq \cdots \ \geq f(p_1) - f(z_1') \ \geq 1$.

3. Proof of Theorem

The necessity part follows by an easy induction on the number of admissible operations defining the coil and the fact that any stable tube satisfies the conditions (c1)-(c4) in Theorem. To prove the sufficiency part, we need a number of lemmas.

Assume Γ satisfies the conditions (c1)-(c4) in Theorem. We need some facts from section 4.2 in [4]. The set of points which are the starting, or ending, points of a mesh in Γ with a unique middle term is called the mouth of Γ. Thus an exceptional mesh must have one of its middle terms on the mouth or projective-injective.

Lemma 3.1. *Let p be an exceptional projective, x be the unique direct predecessor of p. Then the ray $[x,\ \infty[$ exists.*

Proof. By the condition $(c2)$, the ray $[p,\ \infty[$ exists, denoted by

$$p = b_0 \to b_1 \to \ \cdots \to b_n \to \cdots\ .$$

If the ray $[x,\ \infty[$ does not exist, then there is a number m such that b_m is projective. It contradicts $(c2)$. The proof is finished.

Remark. Dually, we have that for the direct successor of an exceptional injective, the coray ending at this vertex exists.

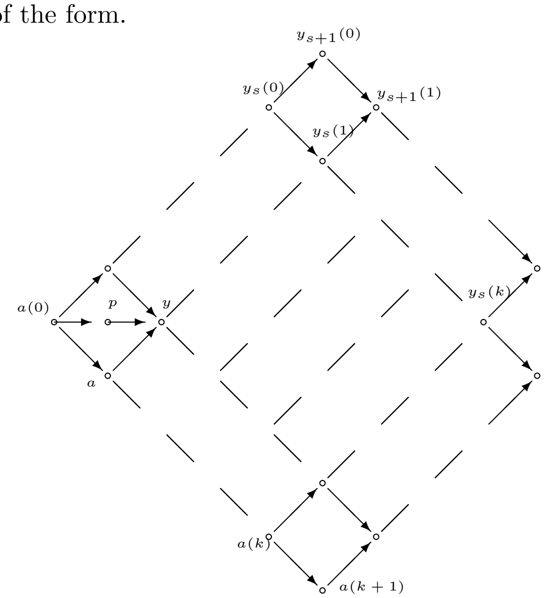
Lemma 3.2. *Let p be an exceptional projective vertex, x be the direct predecessor of p, and $y = \tau^{-n}p$ be such that one of the direct predecessor of y is injective. Denote by a the intersection of $[x,\ \infty[$ with $]\infty,\ y]$, by $a \to y_1 \to \cdots \to y_n(= y) \to y_{n+1} \to \cdots \to y_m$ the sectional path of maximal length from a to mouth. Then the ray $[y_i,\ \infty[$ exists, for each i.*

Proof. The proof is divided into two cases.

1. The vertex p is projective-injective. In this case, $y_1 = y_n = y$ and the ray $[y,\ \infty[$ exists. We assume that the rays $[y_j,\ \infty[$, for all $1 \le j \le s$, exist, and prove that ray $[y_{s+1},\ \infty[$ exists. Otherwise, the ray $[y_s,\ \infty[$ contains an injective. Denote by

$$y_s = y_s(1) \to y_s(2) \to \cdots\ \to y_s(i) \to \cdots$$

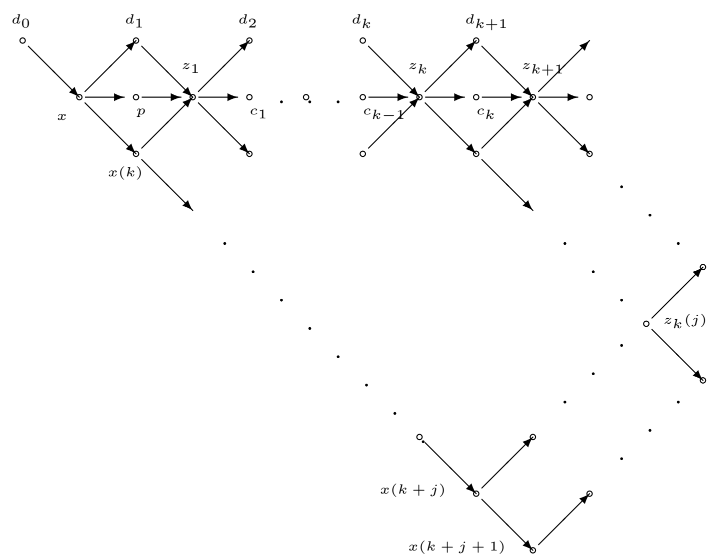
the ray $[y_s,\ \infty[$. Assume that $y_s(k)$ is injective, then we have a full translation subquiver Δ of the form.



Then by Lemma 2.1, we have the equality $f(a(k+1)) + f(y_{s+1}(0)) + f(p) + \delta_{\diamondsuit_{xy_{s+1}(0)a(k+1)_-}} = f(a(0)) - 1$. Thus, by using $f(p) = f(a(0)) + 1$, we get $f(a(k+1)) + f(y_{s+1}(0)) + \delta_{\diamondsuit_{xy_{s+1}(0)a(k+1)_-}} + 2 = 0$. It is a contradiction.

2. The vertex p is projective but not injective. Set $z_i = \tau^{-i}x$, for all $1 \leq i \leq n$. Therefore $z_n = y = y_n$. Let $c_i = \tau^{-i}p,\ 1 \leq i \leq n-1$. Then c_{n-1} is an injective direct predecessor of y, denoted by q.

First, we prove that the rays $[z_i,\ \infty[$, for all $1 \leq i \leq n$, exist. Since the ray $[z_1,\ \infty[$ exists, we shall prove that the ray $[z_{k+1},\ \infty[$ exists under the assumption that the rays $[z_i,\ \infty[$ exist, for all $1 \leq i \leq k$, where $1 \leq k \leq n-1$. Otherwise, the ray $[z_k,\ \infty[$ contains an injective vertex, which is denoted by $z_k(j)$. Then Γ contains a full translation subquiver Δ of the form



with complete meshes ending at z_i or c_i, where $1 \leq i \leq k+1$. By Lemma 2.1, we get

$$f(x(k)) - 1 = f(x(k+j+1)) + f(c_k) + f(d_{k+1}) + \delta_{\diamondsuit_{x(k)d_{k+1}x(k+j+1)_-}}$$
$$f(d_0) + f(d_1) + \cdots + f(d_{k+1}) = f(x(k)) + f(p) + f(c_1) + \cdots + f(c_{k-1}) + \delta_{\Delta_{d_0x(k)d_{k+1}}}.$$

It follows that

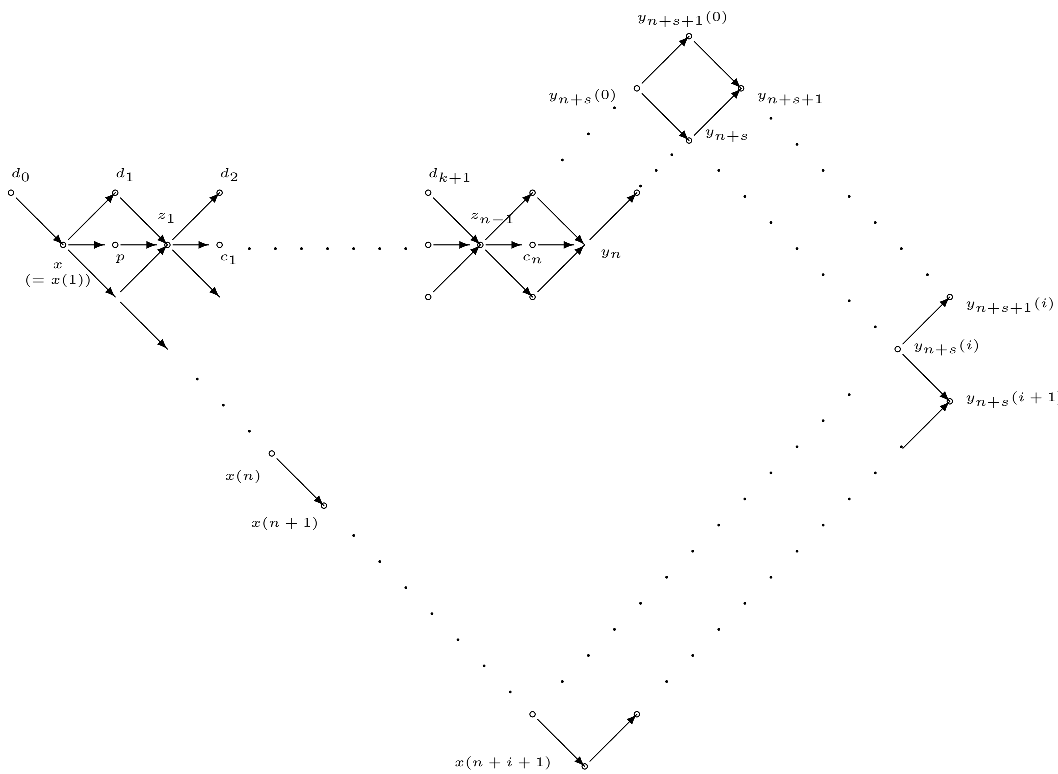
$$f(p) + f(c_1) + \cdots + f(c_{k-1}) + f(c_k) + f(x(k+j+1)) + \delta_{\diamondsuit_{x(k)d_{k+1}x(k+j+1)_-}}$$
$$+\delta_{\Delta_{d_0x(k)d_{k+1}}} = f(d_0) + f(d_1) + \cdots + f(d_k) - 1.$$

By using

$$\begin{array}{lcl} f(d_0)+f(d_1) & = & f(x)+\delta_{d_1}=f(p)-1+\delta_{d_1} \\ f(d_2)+f(d_3) & = & f(z_2)+\delta_{d_3}=f(c_1)+f(c_2)+\delta_{d_3} \\ & \cdots\cdots & \\ f(d_k)+f(d_{k+1}) & = & f(z_k)+\delta_{d_{k+1}}=f(c_{k-1})+f(c_k)+\delta_{d_{k+1}} \quad (2\mid k) \\ f(d_{k-1})+f(d_k) & = & f(z_{k-1})+\delta_{d_k}=f(c_{k-2})+f(c_{k-1})+\delta_{d_k} \quad (2\nmid k), \end{array}$$

we obtain that $f(x(k+j+1))+2<0$. It is a contradiction.

Secondly, we prove that the rays $[y_i,\ \infty[$ exist, for $i\geq n+1$. We assume that the rays $[y_{n+i},\ \infty[$ exist, for all $i\leq s\leq m$. We shall prove that the ray $[y_{n+s+1},\ \infty[$ exists. Indeed otherwise the ray $[y_{n+s},\ \infty[$ contains an injective vertex, denoted by $y_{n+s}(i)$. Then there is a full translation subquiver of Γ of the form



with complete meshes at $z_i,\ c_i$ for all i.

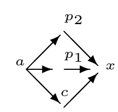
If $s>1$, then c_n is injective, it follows that n is odd, and $f(c_n)-f(x_n)=1$ by Lemma 2.3. By Lemma 2.1, we have that $f(x(n))-1=f(x(n+i+1))+f(y_{n+s+1}(0))+f(c_n)+\delta_{\diamondsuit_{x(n)y_{n+s+1}(0)x(n+i+1)_-}}$. Then we have $f(x(n+i+1))+f(y_{n+s+1}(0))+\delta_{\diamondsuit_{x(n)y_{n+s+1}(0)x(n+i+1)_-}}+2=0$. It is a contradiction. The proof is finished.

Remark. One can obtain dually the statment about the exstence of corays under the condition of Lemma 3.2 .

Remark. The proof of the lemma above provides a proof of the statement that there does not exist any vertex in any ray $[y_i, \infty[$, which is a direct predecessor of a projective-injective vertex.

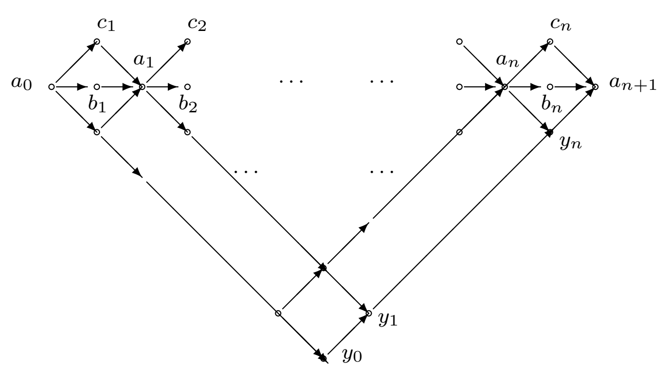
Lemma 3.3. *The middle term of any mesh with three middle terms in* Γ *contains at most one projective vertex and at most one injective vertex, but not two such vertices.*

Proof. Assume that Γ contains a mesh with three middle terms, two of which are projective or injective.



If one is projective, another is injective, then there is a mesh with p_2 being projective, p_1 being injective Then $f(p_1) = f(x) + 1,\ f(p_2) = f(a) + 1,\ f(a) + f(x) = f(p_1) + f(p_2) + f(c)$. It follows that $f(c) + 2 = 0$, and this is a contradiction.

If the two of the middle terms of the mesh are projective, none of them is injective, then there is a full translation subquiver of Γ with complete meshes at each vertex except c_i and b_i for all $i \geq 2$, with b_1 and c_1 being projective, b_n or c_n being injective,



Then $f(b_1) = f(c_1) = f(a_0) + 1$, which follows that $\delta_{c_2} = 0$ and $f(c_2) = f(b_2)$. By an easy induction on i, one can get that $\delta_{c_i} = 0$ and $f(b_i) = f(c_i)$ for all i, in particular, $f(b_n) = f(c_n) = f(a_{n+1}) + 1$. By Lemma 2.1, we have the equalities

$$\begin{array}{rcl} f(y_0) + f(b_1) + f(c_1) & = & f(a_0) + f(y_1) \\ f(y_1) + f(b_2) + f(c_2) & = & f(a_1) + f(y_2) \\ & \ldots\ldots & \\ f(y_n) + f(b_n) + f(c_n) & = & f(a_n) + f(a_{n+1}). \end{array}$$

Then by adding up each side of all equalities above, we obtain

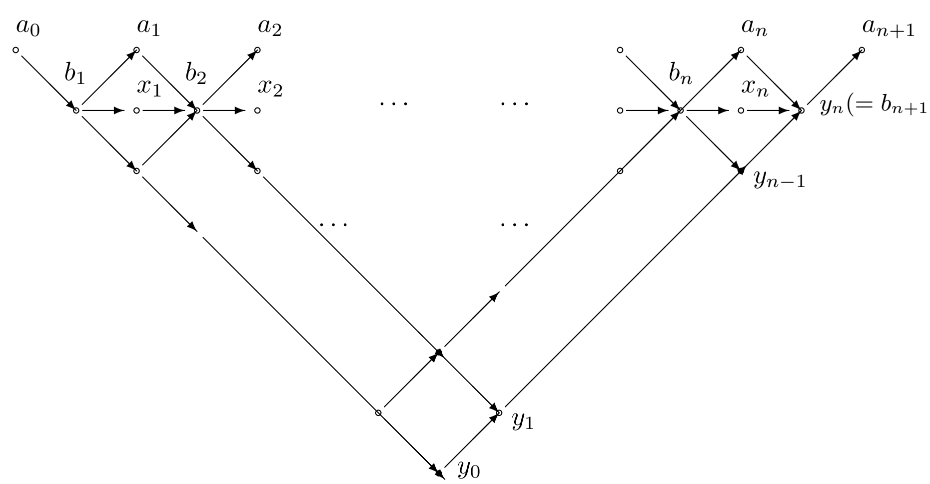
$$f(y_0) + f(b_1) + \cdots + f(b_n) + f(c_1) + \cdots + f(c_n) = f(a_0) + f(a_1) + \cdots + f(a_{n+1}).$$

We note that $f(a_i)+\delta_{b_{i+1}} = f(b_i)+f(b_{i+1}),\ f(b_i) = f(c_i),\ 1 \le i \le n,\ f(b_1) = f(a_1)+1,\ f(b_n) = f(a_{n+1})+1$. Then we have $f(y_0)+2 \le 0$. This contradiction finishes the proof.

The next lemma shows that the condition (c2) in Theorem 4.2 [4] may be removed with the help of the lemmas above.

Lemma 3.4. *For any mesh with three middle terms, none of which is projective-injective, two of the middle terms must lie on the mouth of* Γ.

Proof. We assume that Γ contains a mesh with three middle terms, none of which is projective-injective. Then Γ contains a full translation subquiver of the form



with x_1 being projective, a_n or x_n being injective, and with complete meshes ending at $b_i,\ x_i$ for all $i > 1$:

Assume the assertion in the lemma is false, this means that there is a $t \in \{2, \cdots, n\}$ such that $\delta_t \neq 0$, where $\delta_i = f(a_i) + f(a_{i-1}) - f(b_i)$. Then we have inequalities $\delta_t + \delta_{t+1} \ge f(a_t)$ and $\delta_{t-1} + \delta_t \ge f(a_{t-1})$. We will prove the first inequality, the proof for the second is similar. For $\delta_t \neq 0$, then beside b_t there is other predecessor of a_t, say z. If z is injective, then $\delta_t \ge f(z) > f(a_t)$, hence $\delta_t + \delta_{t+1} \ge f(a_t)$. If z is not injective, then $\delta_t + \delta_{t+1} \ge f(\tau^{-1}z) + f(z) \ge f(a_t)$. The proof for the inequality $\delta_t + \delta_{t+1} \ge f(a_t)$ is finished. We note here that if n is odd, then x_n is injective; if n is even, then a_n is injective by Lemma 2.3. In the following, we assume that n is odd, then x_n is injective. The proof for n even is similar.

By applying repeatedly Lemma 2.1 to the subquiver above, one obtains the following equalities:

$$\begin{array}{rcl} f(y_0)+f(x_1)+\delta_1+\delta_{\diamondsuit_{b_1a_1y_0y_1}} & = & f(a_0)+f(y_1) \\ f(y_1)+f(x_2)+\delta_2+\delta_{\diamondsuit_{b_2a_2y_1y_2}} & = & f(a_1)+f(y_2) \\ & \cdots\cdots & \\ f(y_{n-1})+f(x_n)+\delta_n+\delta_{\diamondsuit_{b_na_ny_{n-1}y_n}} & = & f(a_{n-1})+f(y_n) \\ f(y_n)+\delta_{n+1} & = & f(a_n)+f(a_{n+1}). \end{array}$$

Then by adding up each side of all equalities above, we obtain an equality:

$$\begin{array}{l} f(y_0)+f(x_1)+f(x_2)+\cdots+f(x_n)+\sum_1^{n+1}\delta_i+\sum_{t=1}^n\delta_{\diamondsuit_{b_ta_ty_{t-1}y_t}} \\ =f(a_0)+f(a_1)+\cdots+f(a_n)+f(a_{n+1}). \end{array}$$

By using that $f(x_1)\geq f(b_1)+1,\ f(x_n)\geq f(b_{n+1})+1,\ f(b_1)+\delta_1=f(a_0)+f(a_1),\ f(b_{n+1})+\delta_{n+1}=f(a_n)+f(a_{n+1})$, we obtain an inequality which is denoted by (*):

$$\begin{array}{l} f(y_0)+2+f(x_2)+f(x_3)+\cdots+f(x_{n-1})+\sum_2^n\delta_i+\sum_{t=1}^n\delta_{\diamondsuit_{b_ta_ty_{t-1}y_t}} \\ \leq f(a_2)+f(a_3)+\cdots+f(a_{n-1}). \end{array}$$

The remaining proof will be divided into two cases as follows:

(1). t is even. By Lemma 2.1, we have equalities

$$\begin{array}{rcl} f(a_{n-1})+f(a_{n-2}) & = & f(b_{n-1})+\delta_{n-1}=f(x_{n-1})+f(x_{n-2})+\delta_{n-1}-\delta_{n-1} \\ & \cdots\cdots & \\ f(a_{t+2})+f(a_{t+1}) & = & f(b_{t+2})+\delta_{t+2}=f(x_{t+2})+f(x_{t+1})+\delta_{t+2}-\delta_{t+2} \\ f(a_{t-1})+f(a_{t-2}) & = & f(b_{t-1})+\delta_{t-1}=f(x_{t-1})+f(x_{t-2})+\delta_{t-1}-\delta_{t-1} \\ & \cdots\cdots & \\ f(a_3)+f(a_2) & = & f(b_3)+\delta_3=f(x_3)+f(x_2)+\delta_3-\delta_3. \end{array}$$

Thus by adding up each side of the equalities above, one obtains an equality

$$\begin{array}{l} \sum_2^{t-1}f(x_i)+\sum_{t+1}^{n-1}f(x_i)+\delta_3+\delta_5+\cdots+\delta_{t-1}+\delta_{t+2}+\cdots+\delta_{n-1} \\ =\sum_2^{t-1}f(a_i)+\sum_{t+1}^{n-1}f(a_i)+\sum\delta_j. \end{array}$$

This, combining with the inequality (*), implies that

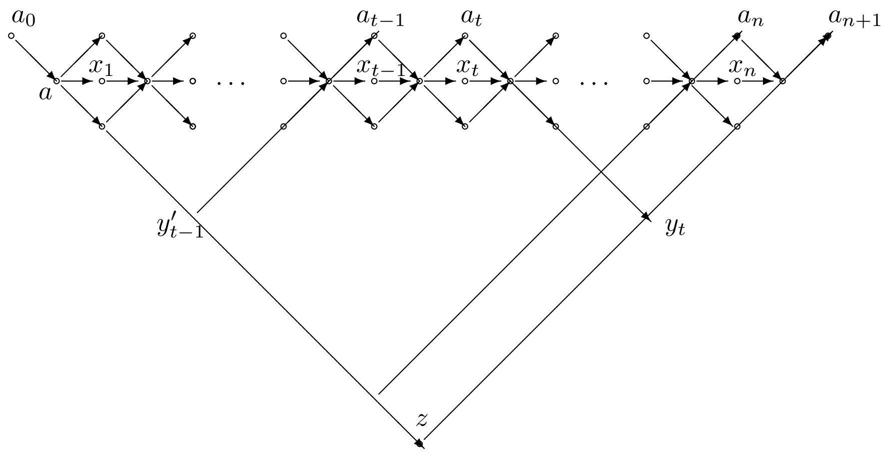
$$f(y_0)+2+\delta_t+\delta_{t+1}\leq\ f(a_t),$$

which is a contradiction, since $\delta_t+\delta_{t+1}\geq\ f(a_t)$.

(2). t is odd. We can also obtain a contradiction in a similar way.

The proof is finished.

Lemma 3.5. Γ *does not contain a full translation subquiver of the form*



with $\delta_t \geq max\{f(a_t)+1, f(a_{t-1})+1\}$, *with* x_1 *being projective and* x_n *or* a_n *being injective, and with complete-meshes ending at vertices in* $\Delta_{a_0 z a_{n+1}}$ *except* $a_i,\ i=0,\ 1,\ \cdots,\ n+1$.

Proof. Suppose Γ contains a full translation subquiver of the form shown in the lemma. By Lemmas 2.2, 2.3, 3.4, 3.5, 3.6, we have $f(x_{t-1}) - f(y'_{t-1}) \geq 1$, and $f(a_t) - f(y_t) \geq 1$ or $f(a_{t-1}) - f(y'_{t-1}) \geq 1$, and $f(x_t) - f(y_t) \geq 1$. But then we obtain $f(y'_{t-1}) + f(y_t) = f(z) + f(x_{t-1}) + f(x_t) + f(c)$, by Lemma 2.1. This combined with $f(c) = f(a_t)+1 = f(a_{t-1})+1$, implies that $f(z)+f(x_t)+2 \leq 0$ or $f(z) + f(x_{t-1}) + 2 \leq 0$, which is a contradiction.

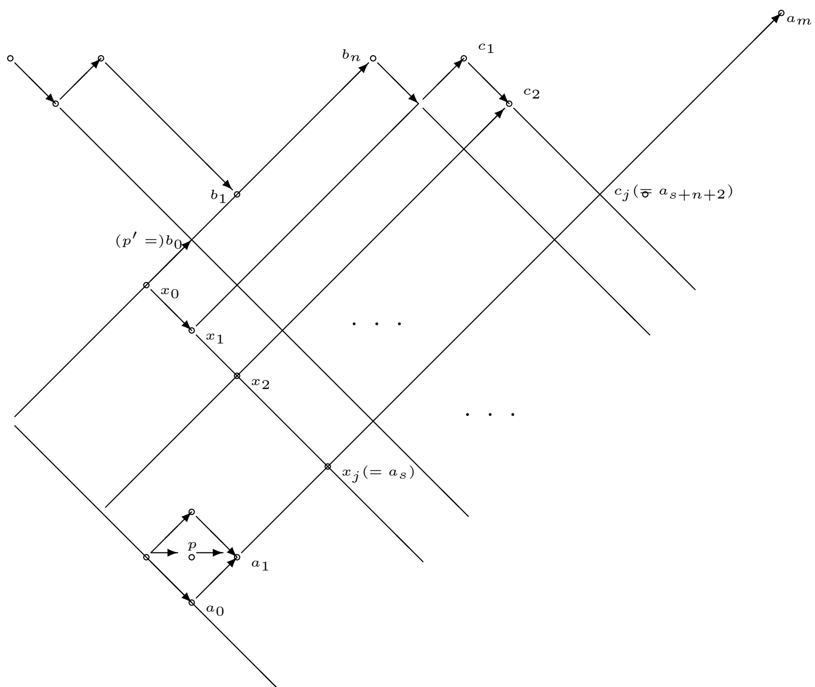
Lemma 3.6. *The mesh category* $k(\Gamma)$ *contains no oriented cycle of projectives*

Proof. The lemma can be proved in a similar way as Lemma 4.5 in [4] with the help of lemmas above.

Proof of Theorem: It remains to prove the sufficiency part. We prove it by induction on the number of projective vertices in Γ. If Γ does not contain any exceptional projective vertex, then Γ is a tube. It implies that Γ is a coil obtained from a stable tube by some admissible operations of type ad1) or ad1)* [7].

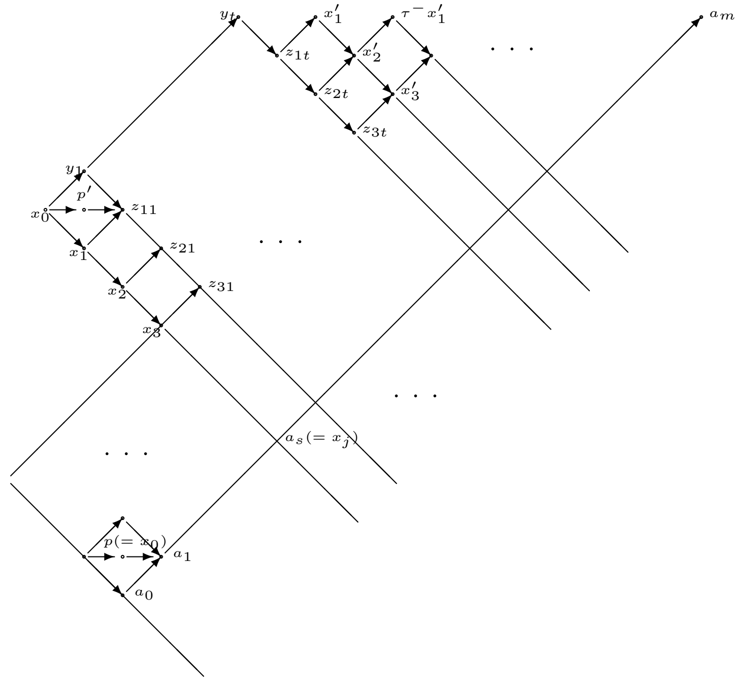
We assume that Γ contains an exceptional projective. First we should remark that Γ contains a (maximal) tube as a cofinite full translation subquiver [4]. Therefore there are only finitely many projective vertices and injective vertices in Γ. It follows from Lemmas 3.6, 3.1 that there is an exceptional projective p with the following property: if we let a be the direct predecessor of p, let $c = \tau^{-n}x$ be the vertex such that one of the direct predecessor of c is injective, then there is no projective vertex such that the direct predecessors of it are on the ray $[a_m, \infty[$. Where $a_0 = [a,\ \infty[\quad]\infty,\ c]$, and $a_0 \to a_1 \to \cdots \to a_n \to\ a_{n+1} \to \cdots \to a_m$ is the sectional path with maximal length from a_0 to a mouth point. Let s be the largest index such that there exists a ray passing through a_s, and the ray contains a direct predecessor of some projective vertex p'. Then by the lemmas above, Γ admits a mesh-complete subquiver Δ of one of the forms below, which depends on p and p' :

(1). If p is a projective-injective vertex and p' is an ordinary projective vertex, then Δ is of the form



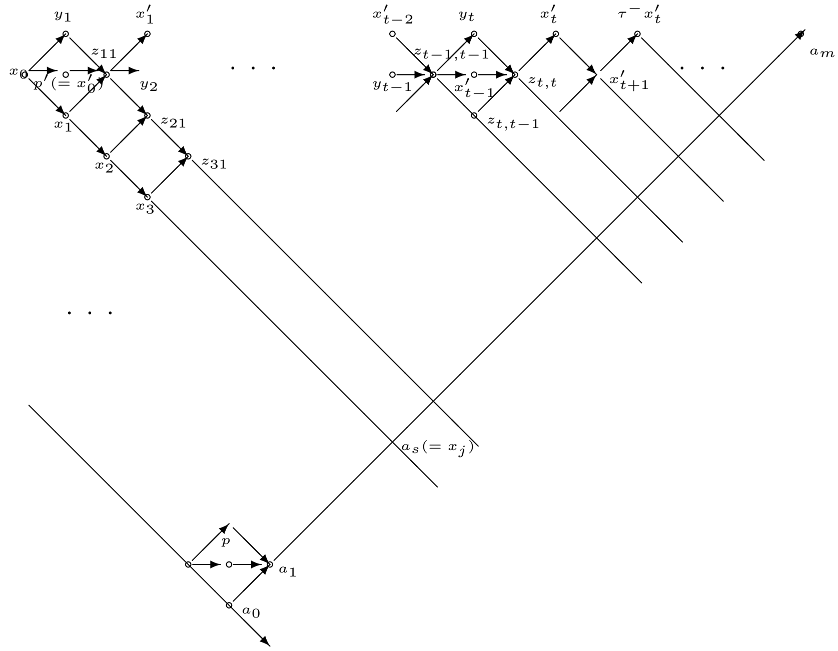
Denote by R the set of points in Γ lying on the sectional paths from the mouth to infinity passing through $b_0, \cdots, b_n$, and by Γ_1 the translation quiver obtained from Γ by deleting R and replacing the sectional paths $x_i \to \cdots \to c_i$ (if they exist) by arrows $x_i \to c_i,\ i \geq 1$. Then Γ_1 still satisfies the conditions (c1)-(c4) and has at least one projective vertex less than Γ.

(2). If p and p' are projective-injective, then Δ is of the form:

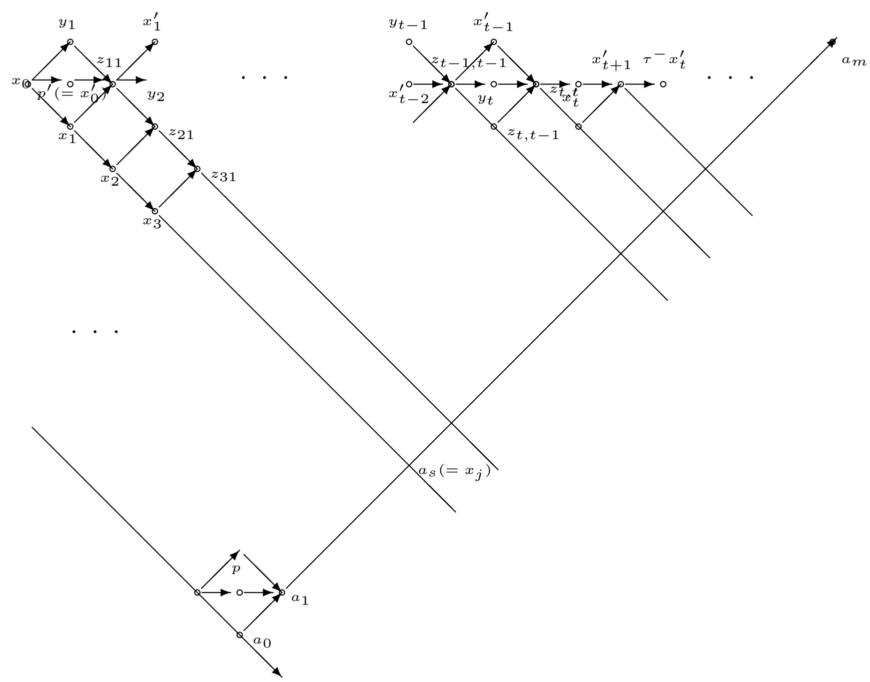


Denote by R the set of vertices P', x'_i, $i \geq 0, z_{ij}$, $i \geq 1$, $1 \leq j \leq t$ and by Γ_1 the translation quiver obtained from Γ by deleting R and replacing the sectional paths $x_i \to \cdots \to \tau^{-1}x'_{i-1}$ (if they exist) by arrows $x_i \to \tau^{-1}x'_{i-1}$. Then Γ_1 still satisfies the conditions (c1)-(c4) and has one projective less than Γ.

(3). If p is a projective-injective vertex and p' is projective, not injective, then Δ is of the form:



if t is odd, in this case x'_{t-1} is injective, or



if t is even, and in the case x_{t-1} is injective. Denote by R the set of vertices $x'_i,\ i \geq 0,\ z_{ij},\ i \geq 1,\ 1 \leq t$, and by Γ_1 the translation quiver obtained from Γ by deleting R, and replacing the sectional paths $y_i \to\ \cdots\ \to y_{i+1}$ by arrows $x_i \to y_{i+1}(0 \leq i \leq t-1)$, the sectional paths $y_i \to z_{ii} \to y_{i+1}$ by arrows $y_i \to y_{i+1}$ $(1 \leq i \leq t-1)$ and the sectional paths $x_i \to \cdots \to x'_i \to \tau^{-1}x'_{i-1}$ (if they exist) by arrows $x'_i \to \tau^{-1}x'_{i-1},\ i \geq t+1$. Clearly, Γ_1 satisfies the conditions (c1)-(c4).

(4).If p is a projectivevertex, not injective vertex and p' is an ordinary projective vertex, the proof is similar to (1).

(5).If p is projective, not injective and $p' = p$, the proof is similar to (3).
For each case, we obtain a translation quiver Γ_1, which satisfies the conditions (c1)-(c4) and has one projective vertex less than Γ. Then an induction on the number of projective vertices finishes the proof.

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