Cluster algebras arising from cluster tubes [1]

Yu Zhou and Bin Zhu

Department of Mathematical Sciences
Tsinghua University
100084 Beijing, P. R. China
E-mail: yu-zhou06@mails.tsinghua.edu.cn

Department of Mathematical Sciences
Tsinghua University
100084 Beijing, P. R. China
E-mail: bzhu@math.tsinghua.edu.cn

Abstract

We study the cluster algebras arising from cluster tubes with rank bigger than 1. Cluster tubes are 2−Calabi-Yau triangulated categories which contain no cluster tilting objects, but maximal rigid objects. Fix a maximal rigid object T in the cluster tube Γ_n of rank n ($n > 1$). For any indecomposable rigid object M in Γ_n, we define an analogous X_M of Caldero-Chapton's formula (or Palu's cluster character formula) by using the geometric information of M. We show that $X_M, X_{M'}$ satisfy the mutation formula for cluster variables when M, M' form an exchange pair, and that $X_? : M \mapsto X_M$ gives a bijection from the set of indecomposable rigid objects in Γ_n to the set of cluster variables of the cluster algebra of type C_{n-1}, which induces a bijection between the set of (basic) maximal rigid objects in Γ_n and the set of clusters. This strengths a surprising result proved recently by Buan-Marsh-Vatne [BMV] that the combinatorics of maximal rigid objects in the cluster tube Γ_n encodes the combinatorics of the cluster algebra of type B_{n-1} since the combinatorics of cluster algebras of type B_{n-1} and of type C_{n-1} are the same by one of results of Fomin-Zelevinsky in [FZ2]. As a consequence, we give a categorification of cluster algebras of type C.

Key words. Cluster tubes; Maximal rigid objects; Cluster variables; cluster algebras of type C.

Mathematics Subject Classification. 16G20, 16G70, 13F60.

1 Introduction

Cluster algebras were introduced around 2000 by Fomin-Zelevinsky [FZ1] in order to give an algebraic and combinatorial framework for the canonical basis of quantum groups and for the notion of total positivity for semisimple algebraic groups, see [FZ4][F] for nice surveys on this topic and its background. Since they introduced, interesting connections between such algebras and several branches of mathematics have emerged. In the categorification theory of cluster algebras, cluster categories [BMRRT, CCS, Ke1, CC, Am, Du1], and (stable) module categories over preprojective algebras [GLS1, GLS2, BIRS] play a central role. They all have cluster tilting objects, and those cluster tilting objects model the clusters of the corresponding cluster algebras via Caldero-Chapoton's formula [CC] in the case of cluster categories or Geiß-Leclerc-Shröer's map [GLS1] in the case of preprojective algebras. This motivates the study of arbitrary 2−Calabi-Yau triangulated categories with cluster tilting objects (subcategories). Palu defined the cluster character for the 2−CY triangulated categories which have cluster tilting objects [Pa1] (see also [FK]). Recently Plamondon defined cluster character for 2−Calabi-Yau triangulated categories with infinite dimensional morphism spaces [Pla1-2]. It was proved in [IY] [BIRS] that one can mutate cluster tilting object $T = \oplus_{i=1}^{n} T_i$ (respectively maximal rigid object) at any indecomposable direct

[1] Supported by the NSF of China (Grants 11071133)

summand T_i to get a new cluster tilting object $\mu_i(T)$ (resp. maximal rigid object) via exchange triangles in a $2-$CY triangulated category Γ. To any maximal rigid object T, one can associate an integer matrix A_T by using the exchange triangles (where A_T we define is the transpose of B_T defined in [BMV], see Section 2 for the precise definition please). If A_T and $A_{\mu_i(T)}$ are related by Fomin-Zelevinsky's matrix mutation for any maximal rigid object T and any direct summands T_i, then we call that the maximal rigid objects form a cluster structure in Γ [BMV, BIRS]. In [BMV], Buan-Marsh-Vatne showed that maximal rigid objects without $2-$cycles form a cluster structure in Γ (see also [BIRS]). Therefore cluster tilting objects and maximal rigid objects are important objects in $2-$CY triangulated categories. They have many nice properties, see for example [I], [KR], [KZ], [IY], [DK], [V], [Y], [ZZ]. It was proved that any maximal rigid object in $2-$Calabi-Yau triangulated categories admitting a cluster tilting object is cluster tilting [ZZ]. Cluster tilting objects are obvious maximal rigid, but the converse is not true, see [BIKR] for the first examples. Cluster tubes provide the second examples, in which the quivers of endomorphism algebras of maximal rigid objects contain loops, but no $2-$cycles [BMV]. Cluster tubes of rank n, denoted by Γ_n, is by definition, the orbit category by $\tau^{-1}[1]$ of the derived category of the hereditary abelian category of nilpotent representations of the quiver with underlying graph $\widetilde{A}_{n-1}$ and with cyclic orientation. It is a $2-$CY triangulated category [Ke][BKL1-2]. In [BMV], a classification of maximal rigid objects in the cluster tube Γ_n is given. The maximal rigid objects are proved to form a cluster structure. Furthermore they use the geometric description of the exchange graph of the cluster algebras of type B_{n-1} in [FZ2] to prove that there is a bijection between the set of indecomposable rigid objects in the cluster tube Γ_n and the set of cluster variables of the cluster algebra of type B_{n-1}. Under this bijection, maximal rigid objects go to clusters [BMV]. Since the cluster combinatorics of the cluster algebras of type C_{n-1} is the same as that of the cluster algebras of type B_{n-1} by Proposition 3.15 in [FZ2], there is also a bijection between the set of indecomposable rigid objects in the cluster tube Γ_n and the set of cluster variables of the cluster algebra of type C_{n-1}.

The aim of the paper is to study the cluster algebras arising from cluster tubes. This is the first attempt to the well-known question how to define cluster character with respect to a maximal rigid object in a $2-$Calabi-Yau triangulated category, in which maximal rigid objects may have loops (compare to [Pl1-2]). We give an analogue of Caldero-Chapoton's formula [CC] (or Palu's character [Pa1]) for cluster tubes. Fix a basic maximal rigid object T in the cluster tube $\Gamma_n (n > 1)$. A_T denotes the skew-symmetrizable matrix associated with T, which is of type C_{n-1} [BMV] (please see the precise meaning in Section 2). For any indecomposable rigid object M in Γ_n, with respect to T, we define an analogous X_M of the Caldero-Chapoton's formula defined for cluster categories in [CC], see also [Pa1]. We prove that the formula X_M satisfies mutation formula for cluster variables: i.e. if M and M^* are indecomposable rigid objects such that $M \oplus N$ and $M^* \oplus N$, for some rigid object N, are maximal rigid objects in Γ_n, then $X_M \cdot X_{M^*} = X_E + X'_E$ where E, E' are the middles of the exchange triangles: $M \to E \to M^* \to M[1]$, $M^* \to E' \to M \to M^*[1]$. We note here that the dimension of $\mathrm{Ext}^1(M, M^*)$ can be 2 (compare to the cases considered before in [CC],[Pa1],[FK],[Pla1-2], the $k-$dimension of $\mathrm{Ext}^1(M, M^*)$ is always one). Thus $X_?$ gives a bijection from the set of indecomposable rigid objects in Γ_n to the set of cluster variables of the cluster algebras of type C_{n-1}. This gives an explicit bijection parallel with that given by Buan-Marsh-Vatne [BMV] for type B_{n-1} (since there is a natural bijection between type B_{n-1} and type C_{n-1}, see [FZ2]). The algebra generated by the X_M, where M runs over indecomposable rigid objects in Γ_n is isomorphic to the cluster algebras of type C_{n-1}. In [Du2], Dupont proved the multiplication formula for cluster characters associated to regular modules over the path algebra of any representation-infinite quiver; Ding-Xu [DX] also defined an analogous map for cluster tubes and gave multiplication formulas. But their formulas are not the exchange formula for cluster variables on the one hand, and their maps can not be used to realize the cluster algebra of type C or B on the other hand.

The paper is organized as follows: In Section 2, we recall some basics on cluster algebras and 2−CY triangulated categories. In particular, we recall the definition of cluster tubes and basic description on indecomposable rigid objects in cluster tubes [BMV]. In Section 3, for any positive number $n > 1$, fix a basic maximal rigid object T in Γ_n, We calculate the index of any indecomposable rigid object M respect to T (defined in [Pa1, DK, Pla]) and define the analogue X_M of CC-maps or Palu's map for indecomposable rigid object M with respect to T. This map $X_?$ is called simply cluster map. Using the structure of the cluster tube Γ_n of rank n, we divide the set of indecomposable rigid objects into three disjoint subsets. Using the structure of endomorphism algebras of T [BMV, V,Y, ZZ], we calculate the explicit formula of X_M according to which subset M belongs. In Section 4, we prove that X_M, X_{M^*} satisfy the mutation formula when M, M^* form a mutation pair. Using the mutation triangles, we explain the matrix A_T associated with T is a skew-symmetrizable matrix of type C_{n-1}. We prove that the map $X_?$ gives a bijection between the set of indecomposable rigid objects in Γ_n and set of cluster variables of A_T, which also induces a bijection between of the set of basic maximal rigid objects and the set of clusters of A_T. It follows that the cluster algebra generated by X_M, where M runs over all indecomposable rigid objects, is isomorphic to the cluster algebra of type C_{n-1}. In the final section, we give an application of the cluster map $X_?$. We prove that the simplicial complex generated by the indecomposable rigid objects in Γ_n gives a realization of the cluster complex of the root system of type C_{n-1} defined in [FZ2].

2 Preliminaries

We recall some basic notations on cluster algebras which can be found in the papers by Fomin and Zelevinsky [FZ1-3]. The cluster algebras we deal with in this paper are without coefficients.

Let $\mathcal{F} = \mathbf{Q}(u_1, u_2, \cdots, u_n)$ be the field of rational functions in indeterminates $u_1, u_2, \cdots, u_n$. Set $\underline{u} = \{u_1, u_2, \cdots, u_n\}$. Let $A = (a_{ij})$ be an $n \times n$ skew-symmetrizable integer matrix. For any $k \in [1, n]$, where $[1, n]$ denotes the set of the numbers $1, 2, \cdots, n$, the mutation $\mu_k(A)$ of A in direction k is by definition, an integer matrix $A' = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} -a_{ij} & \text{if } i = k \text{ or } j = k, \\ a_{ij} + \frac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2} & \text{otherwise.} \end{cases}$$

A' is a skew-symmetrizable matrix too.

A seed is a pair $(\underline{x}, A)$, where $\underline{x} = \{x_1, x_2, \cdots, x_n\}$ is a transcendence base of $\mathcal{F}$ and A is an $n \times n$ skew-symmetrizable integer matrix.
A mutation $\mu_k(\underline{x}, A)$ of a seed $(\underline{x}, A)$ in direction k is a new seed $(\underline{x'}, A')$, where $A' = \mu_k(A)$, and $\underline{x'} = (\underline{x} \setminus \{x_k\}) \bigcup \{x'_k\}$, x'_k is defined in the following mutation formula:

$$x_k x'_k = \prod_{a_{ik} > 0} x_i^{a_{ik}} + \prod_{a_{ik} < 0} x_i^{-a_{ik}}.$$

The "mutation equivalence $\approx$" is an equivalence relation on the set of all seeds generated by the mutation.
The cluster algebra $\mathcal{A}_A$ associated to the skew-symmetrizable matrix A is by definition the subalgebra of $\mathcal{F}$ generated by all x_i in $\underline{x}$ such that $(\underline{x}, A_1) \approx (\underline{u}, A)$. Such $\underline{x} = (x_1, x_2, \cdots, x_n)$ is called a cluster of the cluster algebra $\mathcal{A}_A$ or simply of the matrix A, and any x_i is called a cluster variable. The seed $(\underline{u}, A)$ is called an initial seed. The set of all cluster variables is denoted by χ_A. If the set χ_A is finite, then the cluster algebra $\mathcal{A}_A$ is called of finite type. For any skew-symmetrizable integer matrix A, one can define the Cartan counterpart C_A of A as follow: $C_A = (c_{ij})_{n\times n}$, where

$$c_{ij} = \begin{cases} -|a_{ij}| & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

It was proved by Fomin-Zelevinsky [FZ3] that cluster algebras are of finite type if and only if there is a seed $(\underline{x'}, A_1)$ which is equivalent to $(\underline{u}, A)$ such that the Cartan counterpart C_{A_1} of A_1 is a Cartan matrix of finite type. In this case, the type of the Cartan matrix C_{A_1} is called the type of the cluster algebra $\mathcal{A}_A$.

For example if $C_{A_1} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -2 & 2 & -1 & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & & \cdots & -1 & 2 \end{pmatrix}$, then the cluster algebra is called of type C_n.

Now we recall some basics on 2−CY triangulated categories.
Fix an algebraically closed field k. A triangulated category Γ is called $k-$linear provided all Hom-spaces in Γ are $k-$spaces and the compositions of maps are $k-$linear. The $k-$linear triangulated categories in this paper will be assumed Hom-finite and Krull-Remak-Schmidt, i.e. $\dim_k \mathrm{Hom}(X, Y) < \infty$ for any two objects X and Y in Γ, and every object decomposes into a finite direct sum of objects having local endomorphism rings. For using Hom-infinite triangulated categories to study cluster algebras, we refer the recent paper [Pla1-2].
A $k-$linear triangulated category Γ is called 2−Calabi-Yau (2−CY for short) if there is a bifunctorial isomorphism $\mathrm{Hom}(X, Y) \cong D\mathrm{Hom}(Y, X[2])$ for objects $X, Y \in \Gamma$, where $D = \mathrm{Hom}_k(-, k)$ [K1].
The main examples of 2−CY triangulated categories are the cluster categories of abelian hereditary categories with tilting objects [BMRRT, Ke1]; the Hom-finite generalized cluster categories of algebras with global dimension of at most 2[Am]. Many other examples are the stable categories of Cohen-Macaulay modules [BIKR] [BIRS], cluster tubes [BKL] and some others, please see the survey [K3].
Cluster tilting objects are defined first in cluster categories [BMRRT], which are generalized to arbitrary 2−CY triangulated categories by Keller and Reiten in [KR].

Definition 2.1. *Let T be an object of a 2−CY triangulated category Γ.*

1. *T is called rigid provided $Ext^1(T, T) = 0$.*
2. *T is called maximal rigid provided T is rigid and is maximal with respect to this property, i.e. if $Ext^1(T \oplus M, T \oplus M) = 0$, then $M \in addT$. Where $addT$ denotes the subcategory of Γ consisting of (finite) direct sums of direct summands of T*
3. *T is called cluster-tilting provided for any $M \in \Gamma$, $M \in addT$ if and only if $Ext^1(M, T) = 0$.*

From the definition, any cluster tilting object is maximal rigid, but the converse is not true. It was proved in [BMV] that the cluster tube Γ_n of rank n ($n > 1$) has no cluster tilting objects, but maximal rigid objects. See [BIKR] for more such examples. $2-$CY triangulated categories with cluster tilting objects are important for the categorification of cluster algebras of skew-symmetric matrices, see the survey [K2][Re] and the references there.
For a basic maximal rigid object $T = T_1 \oplus \cdots \oplus T_i \oplus \cdots \oplus T_n$ with all T_j indecomposable. Fix any $i \in [1, n]$, write $T = \bar{T} \oplus T_i$, where $\bar{T} = \oplus_{j \neq i} T_j$. Then there are two non-split triangles:

$$T_i^* \xrightarrow{f_i} E_i \xrightarrow{g_i} T_i \rightarrow T_i^*[1],$$
$$T_i \xrightarrow{f_i'} E_i' \xrightarrow{g_i'} T_i^* \rightarrow T_i[1],$$

where f_i and f_i' are minimal left $\bar{T}-$approximations; g_i and g_i' are minimal right $\bar{T}-$approximations. Furthermore T_i^* is indecomposable and $\bar{T} \oplus T_i^*$ is maximal rigid [IY, BIRS]. Define $\mu_i(T) = \bar{T} \oplus T_i^*$. $\mu_i(T)$ is called the mutation of maximal rigid object T in direction i and the two triangles above are called exchange triangles. It is easy to see that $\mu_i \circ \mu_i(T) = T$.
Let $T = T_1 \oplus \cdots \oplus T_i \oplus \cdots \oplus T_n$ be a basic maximal rigid object in Γ. We define an integer matrix $A_T = (a_{ij})$ as follows:
$a_{ij} = \alpha_{ij} - \alpha'_{ij}$, where α_{ij} denotes the multiplicity of T_i as a direct summand of E_j', α'_{ij} denotes the multiplicity of T_i as a direct summand of E_j. Note that $a_{ii} = 0$.
Our definition of the matrix A_T associated to T is the transpose of the matrix B_T associated to T defined in [BMV]. When the endomorphism algebra of T contains no loops or $2-$cycles, then A_T, B_T are skew-symmetric matrices, and then $A_T = -B_T$. In general the matrices A_T, B_T are sign-skew-symmetric (see Lemma 1.2 in [BMV]).
We also note that

Remark 2.2. *We use our definition of the matrix A_T associated to T to replace the matrix B_T defined in [BMV] since we will prove the exchange formula for our cluster formulas of indecomposable rigid objects in Γ_n ($n > 1$), and will prove the cluster algebras realized by cluster tubes are of type C_{n-1}.*

Let Γ be a $2-$CY triangulated category with maximal rigid objects. Suppose that for all maximal rigid objects $T = T_1 \oplus \cdots \oplus T_i \oplus \cdots \oplus T_n$, E_i and E_i' have no common direct summands for any $i \in [1, n]$. Then $\mu_i(A_T) = A_{\mu_i(T)}$ (equivalent to $\mu_i(B_T) = B_{\mu_i(T)}$, the latter was proved in [BMV]). In this case we say that the maximal rigid objects form a cluster structure in Γ as in [BMV].

In what follows, we will focus on cluster tubes, an special $2-$CY triangulated category.

We will denote the tube of rank n by $\mathcal{T}_n$, where n is always assumed to be greater than 1. One realization of this category is the category of finite-dimensional nilpotent representations over k of the cyclic quiver $\overrightarrow{\Delta}_n$ with n vertices such that arrows are going from i to $i+1$ (taken modulo n). It is a $k-$linear hereditary abelian category with Hom-finite, i.e. $\dim\mathrm{Hom}_{\mathcal{T}_n}(X, Y) < \infty$. Each indecomposable representation is uniserial, i.e. it has a unique composition series, and hence is determined by its socle and its length up to isomorphism. We denote by (a, b) in $\mathcal{T}_n$ the unique indecomposable object with socle (a,1) and quasi-length b, where $(a, 1)$ is the simple representation at vertex a, where $a \in [1, n]$ (see Fig.2 in Section 3). $\mathcal{T}_n$ has Auslander-Reiten sequences, and the Auslander-Reiten translation τ is an automorphism of $\mathcal{T}_n$, $\tau(a, b) = (a - 1, b)$.

The cluster tube of rank n is defined in [BKL], see also [BMV], as the orbit category $\Gamma_n := D^b(\mathcal{T}_n)/\tau^{-1}[1]$, where [1] is the shift functor of $D^b(\mathcal{T}_n)$. This category is a triangulated category such that the projection $\pi : D^b(\mathcal{T}_n) \to \Gamma_n$ is a triangle functor. It is also $2-$CY [K][BKL].
Γ_n has Auslander-Reiten triangles which induced from ones in $D^b(\mathcal{T}_n)$. It is easy to see the indecomposable objects in $\mathcal{T}_n$ are also indecomposable in Γ_n (via the composite of the inclusion functor $\mathcal{T}_n \hookrightarrow D^b(\mathcal{T}_n)$ with projection $\pi : D^b(\mathcal{T}_n) \to \Gamma_n$) and all indecomposable objects in Γ_n are of this form. So we use the same (a, b) to denote the indecomposable object in Γ_n which is induced from the object (a, b) in $\mathcal{T}_n$. In what follows, we always use $\mathrm{Hom}(X, Y)$ to denote $\mathrm{Hom}_{\Gamma_n}(X, Y)$ for simplicity.

By the definition of Γ_n, for two objects $X, Y \in \mathcal{T}_n$,

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}_n}(X, Y) \oplus D\mathrm{Hom}_{\mathcal{T}_n}(Y, \tau^2 X).$$

As in [BMV], the maps from X to Y in Γ_n which are from $\mathrm{Hom}_{\mathcal{T}_n}(X, Y)$ are called $\mathcal{T}-$maps and the maps from X to Y in Γ_n which are from $D\mathrm{Hom}_{\mathcal{T}_n}(Y, \tau^2 X)$ are called $\mathcal{D}-$maps.
The indecomposable rigid objects are classified in [BMV]:

Proposition 2.3. *(a, b) is rigid if and only if $b \leq n - 1$.*

We fix $n - 1$ special rigid objects: $(1, n - i)$, denoted by T_i, $i = 1, \cdots, n - 1$. It is easy to see that $\oplus_{i=1}^{n-1} T_i$ is a basic maximal rigid object in Γ_n [BMV]. We will use T to denote this maximal rigid object throughout this paper.
It was proved in [BMV] that the basic maximal rigid objects in Γ_n have no $2-$cycles, and form a cluster structure of Γ_n. Moreover there is a bijection between the indecomposable rigid objects and the cluster variables of the cluster algebra of type B_{n-1}.
For two subcategories $\mathcal{D}_1, \mathcal{D}_2$ of Γ_n, denote by $\mathcal{D}_1 * \mathcal{D}_2$ the full subcategory of Γ_n consisting of object E such that there is a triangle $D_1 \to E \to D_2 \to D_1[1]$, where $D_i \in \mathcal{D}_i$, for $i = 1, 2$.

3 Index and the cluster map $X_?$

Let Γ_n be the cluster tube of rank n and $\mathcal{F} = \mathbf{Q}(x_1, x_2, \cdots, x_{n-1})$ be the field of rational functions in indeterminates $x_1, x_2, \cdots, x_{n-1}$.

In this section, we will define the cluster map $X_?$ from the set of indecomposable rigid objects in Γ_n to $\mathcal{F}$ by using the geometric information of the indecomposable rigid objects. We will give an explicit expression of X_M as a Laurent polynomial of $x_1, \cdots, x_{n-1}$ according to which subset M belongs to, where the set of indecomposable rigid objects in Γ_n is divided into three disjoint subsets.

Fix $T := \oplus_{i=1}^{n-1} T_i$, where $T_i = (1, n - i)$ is as before. T is a maximal rigid object in Γ_n. Let $K_0^{split}(T)$ be the split-Grothendieck group of add$T$, i.e. the free abelian group with a basis $[T_1], [T_1], \cdots, [T_{n-1}]$. We use $[T']$ to denote the image of $T' \in$ add$T$ in $K_0^{split}(T)$, which is also denote the dimension vector with respect to the basis above.

Set $\mathcal{D} = \mathrm{add}T[-1] * \mathrm{add}T$. For an object X of $\mathcal{D}[1]$, there exists a triangle

$$T''_X \to T'_X \xrightarrow{f} X \to T''_X[1]$$

where $T''_X, T'_X \in \text{add}T$. It follows that f is a right add$T-$approximation. We define the index $ind_T(X) = [T'_X] - [T''_X] \in K_0^{split}(T)$ as in [Pa1, DK, Pla]. It was proved in [ZZ] that any rigid object belongs to $\mathcal{D}[1]$ (also in $\mathcal{D}$). The next lemma tells us how to get the right add$T-$approximation of any indecomposable rigid object in Γ_n.

Lemma 3.1. *For any indecomposable rigid object (a, b) in $\mathcal{T}_n$, every right addT$-$approximation of (a, b) in $\mathcal{T}_n$ is a right addT$-$approximation of (a, b) in Γ_n.*

Proof. Let $\varphi : T_0 \to (a, b)$ be a right add$T-$approximation of (a, b) in $\mathcal{T}_n$. We will prove it is a right add$T-$approximation of (a, b) in Γ_n too, i.e. any maps from T_0 to (a, b) factor through φ. We only need to consider $\mathcal{D}-$maps since any maps are the sum of some $\mathcal{T}-$map and some $\mathcal{D}-$map, and $\mathcal{T}$-maps factor through φ for φ is a right add$T-$approximation of (a, b). Now suppose there are non-zero $\mathcal{D}-$maps in $\text{Hom}_{\Gamma_n}(T, (a, b))$. For any non-zero $\mathcal{D}-$map $g : T_i \to (a, b)$, by Lemma 2.2 in [BMV], there exists a $\mathcal{D}-$map $g_1 : T_1 \to (a, b)$ and a $\mathcal{T}-$map $g_2 : T_i \to T_1$ such that $g = g_1 g_2$. It follows from Lemma 2.2 in [BMV] that $\dim\text{Hom}_{\mathcal{T}_n}((a, b), \tau^2 T_1) = 1$. So it is sufficient to show that for any nonzero $\mathcal{D}-$map f in $\text{Hom}_{\Gamma_n}(T_1, T_1)$ and any nonzero $\mathcal{T}-$map h in $\text{Hom}_{\Gamma_n}(T_1, (a, b))$ the composition hf is not zero. The triangle involving f is $(2, 2n-2) \to T_1 \xrightarrow{f} T_1 \xrightarrow{g} (1, 2n-2)$ where g is a $\mathcal{T}-$map. We know that $\alpha g = 0$ for any $\mathcal{T}-$map α in $\text{Hom}_{\Gamma_n}((1, 2n-2), (a, b))$, so h can not factor through g, and then $hf \neq 0$. Thus we prove the assertion. □

We recall the index of an indecomposable rigid object (a, b) respect to T [Pa1, DK, Pla]: $\text{ind}_T(a, b) = [T'] - [T''] \in K_0^{split}(T)$, where T', T'' are the terms in the triangle $T'' \to T' \to (a, b) \to T''[1]$.

Theorem 3.2. *For any indecomposable rigid object (a, b) in Γ_n,*

$$ind_T(a, b) = \begin{cases} [T_1] - [T_{n-a+1}] - [T_{2n-a-b}] & \text{if } a + b \geq n + 1 \\ [T_{n-a-b+1}] - [T_{n-a+1}] & \text{if } a + b \leq n. \end{cases}$$

Proof. For $a = n$, there is a triangle $(1, b) \to 0 \to (n, b) \to (1, b)[1]$, so $\text{ind}_T(n, b) = -[(1, b)]$.
For $1 \leq a \leq n-1$ and $a+b \geq n+1$, there is a minimal right add$T-$approximation $f : T_1 \to (a, b)$ in $\mathcal{T}_n$. Then $\ker(f) = (1, a-1)$ and $\text{coker}(\tau^{-1} f) = (1, a+b-n)$. We get a triangle

$$C \to T_1 \xrightarrow{f} (a, b) \to C[1]$$

in Γ_n and an exact sequence

$$0 \to \text{coker}(\tau^{-1} f) \to C \to \ker(f) \to 0$$

in $\mathcal{T}_n$ (cf. [Y]). Since $\text{Ext}^1_{\mathcal{T}_n}((1, a-1), (1, a+b-n)) = 0$, then $C \cong (1, a-1) \oplus (1, a+b-n)$. Hence $\text{ind}_T(a, b) = [T_1] - [T_{n-a+1}] - [T_{2n-a-b}]$.
For $a + b \leq n$, there is a minimal right add$T-$approximation $f : T_{n-a-b+1} \to (a, b)$ in $\mathcal{T}_n$ with $\ker(f) = (1, a-1)$ and $\text{coker}(\tau^{-1} f) = 0$. Hence $\text{ind}_T(a, b) = [T_{n-a-b+1}] - [T_{n-a+1}]$. □

Let $B = \text{End}_{\Gamma_n}(T[-1])$, then $F := \text{Hom}(T[-1], -) : \mathcal{D}/\text{add}T \to \text{mod}B$ is an equivalence of categories [IY, Y, ZZ], where mod$B$ denotes the category of finite dimensional right $B-$modules.

Definition 3.3. *For any indecomposable rigid object M in Γ_n, we define*

$$X_M = x^{ind_T M} \sum_{e\in\mathbb{N}^{n-1}} \chi\,(Gr_e(Hom(T[-1], M)))\, x^{-\iota(e)}$$

where the sum takes over all dimension vectors e such that there exist an object Y in $\mathcal{D}\cap\mathcal{D}[1]$ with $e = \underline{dim}Hom_{C_n}(T[-1], Y)$; where $\iota(e) = ind_T Y + ind_T Y[1]$ and $\chi\,(Gr_e(Hom(T[-1], M)))$ is the Euler characteristic of $Gr_e(Hom(T[-1], M))$ ([K2]).

The definition of X_M can be extended to rigid objects: for any rigid object $N = \bigoplus_{i=1}^m M_i$ in Γ_n with M_i indecomposable, we define

$$X_N = \prod_{i=1}^m X_{M_i}.$$

In [Pla1-2], similar definition is given respect to a fixed rigid object T in a $2-$CY triangulated category with infinite Hom-spaces. The definition in [Pla1-2] needs an additional assumption that the simple $B-$module at each vertex can be lifted through the functor F to an object in $\mathcal{D}\cap\mathcal{D}[1]$ and then any finite-dimensional $B-$module can be lifted through F to an object in $\mathcal{D}\cap\mathcal{D}[1]$. This assumption is not satisfied in our situation. For example, simple $B-$module S_1 corresponding to T_1 can not lift to any object in $\mathcal{D}\cap\mathcal{D}[1]$ by functor F (see the dimension formula stated before Lemma 3.6). So the definition in [Pla1-2] can not apply to our case. In our definition, we omit the $B-$submodules which can not be lifted through F to $\mathcal{D}\cap\mathcal{D}[1]$.

We give some remarks about the definition:

Remark 3.4. *We will point out in Lemma 3.6 that for an indecomposable rigid object M, if an $B-$submodule of FM can not be lifted to $\mathcal{D}\cap\mathcal{D}[1]$, then so are other $B-$submodules of FM with the same dimension. Then the Euler characteristic is well-defined.*

Remark 3.5. *Why does not $\iota(e)$ depend on the choice of such an Y? We will show this in the proof of Theorem 3.7.*

Following [V][Y], the set of indecomposable objects in $\mathcal{D}[1]$ is the set of indecomposable objects (a, b) satisfying either (1) (a,b) is rigid or (2) $n \leq b \leq 2n-2$ and $a + b \leq 2n-1$. We divide the set of indecomposable objects in $\mathcal{D}$ into five subsets (see Fig.1):

$$\begin{array}{rcl} O & = & \{(a, b) \mid a = 1, b \leq n-1\} \\ I & = & \{(a, b) \mid 2 \leq a \leq n-1, a+b \leq n\} \\ II & = & \{(a, b) \mid a+b \geq n+1, b \leq n-1\} \\ III & = & \{(a, b) \mid a+b \leq 2n-1, a \neq 1, b \geq n\} \\ IV & = & \{(a, b) \mid a+b = 2n, a \neq 1, b \geq n\} \end{array}$$

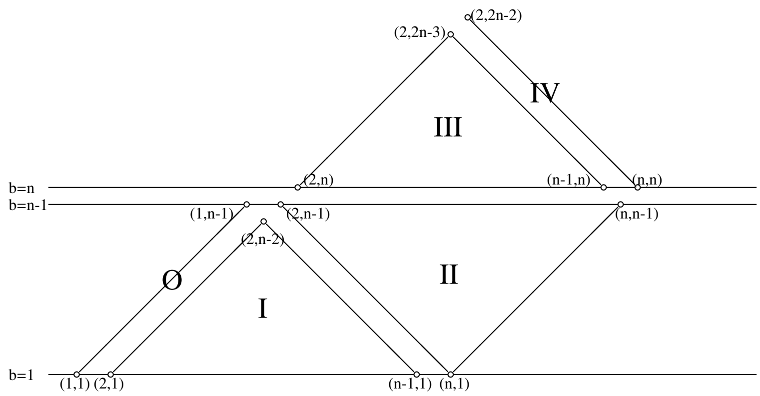

Fig.1 Five disjoint subsets of $\mathcal{D}$.

It is easy to see that for any indecomposable object (a,b) in $\mathcal{D}$, we have

$$\mathrm{dimHom}(T_1[-1],(a,b)) = \begin{cases} 0 & \text{if } (a,b)\in O \text{ or } I; \\ 2 & \text{if } (a,b)\in II \text{ or } III; \\ 1 & \text{if } (a,b)\in IV. \end{cases}$$

So we have the following fact.

Lemma 3.6. *Let M be an indecomposable rigid object in Γ_n and Y be an object in $\mathcal{D}$ such that FY is a $B-$submodule of FM. Then $Y\in\mathcal{D}\cap\mathcal{D}[1]$ if and only if dimHom$(T_1[-1],Y)\neq 1$.*

Proof. For an object Y in $\mathcal{D}$, it is easy to see that $Y\in\mathcal{D}\cap\mathcal{D}[1]$ if and only if all indecomposable summands of Y are in $O\bigcup I\bigcup II\bigcup III$. If $\mathrm{dimHom}(T_1[-1],Y)=2$, then it follows from $\mathrm{dimHom}(T_1[-1],M)\leq 2$ that $\mathrm{dimHom}(T_1[-1],M)=2$ and $\mathrm{Hom}(T_1[-1],Y)=\mathrm{Hom}(T_1[-1],M)$. Then there is a $\mathcal{T}-map$ f in $\mathrm{Hom}(T_1[-1],M)$ such that f is a generator of $B-$module FM. So $FY\cong FM$ and then $Y\cong M\oplus T'$, where $T'\in\mathrm{add}T$. Hence $Y\in\mathcal{D}\cap\mathcal{D}[1]$.
When $\mathrm{dimHom}(T_1[-1],Y)=1$, Y has some summand in IV, then $Y\notin\mathcal{D}\cap\mathcal{D}[1]$.
When $\mathrm{dimHom}(T_1[-1],Y)=0$, all summands of Y are in O or I, then $Y\in\mathcal{D}\cap\mathcal{D}[1]$.
The proof is finished.

□

Let $E_i:\mathrm{Hom}(T_i[-1],Y)\to\mathrm{Hom}(T[-1],Y)$ be the natural embedding and e_i be the dimension vector of the simple $B-$module S_i corresponding to T_i. In the following expression of x_m and $[T_m]$, when $m=n$, x_n is taken to 1 and $[T_m]$ is taken to 0.

Theorem 3.7. *The X_M in Definition 3.3 is well-defined and for any indecomposable rigid object (a,b),*

$$X_{(a,b)} = \begin{cases} x_{n-b} & \textit{if } (a,b)\in O; \\ x_{n-a+2}x_{n-a-b+1}\left(\sum_{k=n-a-b+1}^{n-a+1}\frac{1}{x_kx_{k+1}}\right) & \textit{if } (a,b)\in I; \\ x_1x_{n-a+2}x_{2n-a-b+1}\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{n-a+1}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{l=1}^{2n-a-b}\frac{1}{x_lx_{l+1}}\right)\right) & \textit{if } (a,b)\in II. \end{cases}$$

Proof. For the first statement, we only need to show that for an indecomposable rigid object $M \in \mathcal{D}$, if there are two $B-$submodules FY' and FY'' of FX with the same dimension $\underline{\dim} FY' = \underline{\dim} FY''$, where $Y', Y'' \in \mathcal{D}$, then $\mathrm{ind}_T Y' + \mathrm{ind}_T Y'[1] = \mathrm{ind}_T Y'' + \mathrm{ind}_T Y''[1]$. This and the second statement will be proved in the following detailed calculations. We note that if $FY' \cong FY''$ as a $B-$submodule of FM, then $Y' \oplus T' \cong Y'' \oplus T''$, for some $T', T'' \in \mathrm{add}T$. Hence we have $\mathrm{ind}_T(Y' \oplus T') + \mathrm{ind}_T(Y' \oplus T')[1] = \mathrm{ind}_T(Y'' \oplus T'') + \mathrm{ind}_T(Y'' \oplus T'')[1]$. Since $[T'[1]] = -[T']$ and $[T''[1]] = -[T'']$, it follows that $\mathrm{ind}_T Y' + \mathrm{ind}_T Y'[1] = \mathrm{ind}_T Y'' + \mathrm{ind}_T Y''[1]$. So we can assume that the objects Y', Y'' are those such that FY', FY'' are not isomorphic to each other as submodules of FM in the following.

1. The case of $(a, b) \in O$. In this case, we have $(a, b) \in \mathrm{add}T$, and then $F((a, b)) = Hom(T[-1], (a, b)) = 0$. Hence $\mathrm{ind}_T(a, b) + \mathrm{ind}_T(a, b)[1] = 0$, $\mathrm{ind}_T(a, b) = [T_{n-b}]$, and $X_{(a,b)} = x_{n-b}$.

2. The case of $(a, b) \in I$. It is easy to compute the dimension vector of $F((a, b))$:

$$\dim\mathrm{Hom}(T_i[-1], (a, b)) = \begin{cases} 1 & \text{if } n-a-b+2 \le i \le n-a+1 \\ 0 & \text{otherwise.} \end{cases}$$

   We write $F((a, b)) = \oplus_{j=n-a-b+2}^{n-a+1} E_j(\mathrm{Hom}(T_j[-1], (a, b)))$ (as vector spaces). For any nonzero element f in $\mathrm{Hom}(T_i[-1], (a, b))$, $n-a-b+2 \le i \le n+1-a$, $E_i(f)B = \oplus_{j=i}^{n-a+1} E_j(\mathrm{Hom}(T_j[-1], (a, b)))$. For any object $Y \in \mathcal{D} \cap \mathcal{D}[1]$, if $F(Y)$ is a non-zero $B-$submodule of $F((a, b))$, then we can find non-zero $f \in F(Y)$ such that $f \in \mathrm{Hom}(T_i[-1], (a, b))$ with minimal number i. Obviously $i > 1$ since $\mathrm{Hom}(T_1[-1], (a, b)) = 0$. We have that $F(Y) = E_i(f)B = \oplus_{j=i}^{n-a+1} E_j(\mathrm{Hom}(T_j[-1], (a, b)))$ and $\dim\mathrm{Hom}(T_1[-1], Y) \neq 1$. Hence $FY \cong F((a, n-a-i+2))$ with $e = \underline{\dim} FY = \sum_{k=i}^{n-a+1} e_k$, $n-a-b+2 \le i \le n-a+1$. Then $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = \mathrm{ind}_T(a, n-a-i+2) + \mathrm{ind}_T(a, n-a-i+2)[1] = \mathrm{ind}_T(a, n-a-i+2) + \mathrm{ind}_T(a-1, n-a-i+2) = ([T_{i-1}] - [T_{n-a+1}]) + ([T_i] - [T_{n-a+2}])$ and $\mathrm{ind}_T(a, b) = [T_{n-a-b+1}] - [T_{n-a+1}]$. In this case, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a, b)))) = 1$. Hence, by Definition 3.3, we have that $X_{(a,b)} = \frac{x_{n-a-b+1}}{x_{n-a+1}} \left(1 + \sum_{k=n-a-b+2}^{n-a+1} \left(\frac{x_{n-a+1}}{x_{k-1}} \frac{x_{n-a+2}}{x_k}\right)\right) = x_{n-a+2} x_{n-a-b+1} \left(\sum_{k=n-a-b+1}^{n-a+1} \frac{1}{x_k x_{k+1}}\right)$.

3. The case of $(a, b) \in II$. In this case,

$$\dim\mathrm{Hom}(T_i[-1], (a, b)) = \begin{cases} 2 & \text{if } 1 \le i \le n-a+1; \\ 1 & \text{if } n+2-a \le i \le 2n-a-b; \\ 0 & \text{otherwise.} \end{cases}$$

   We write $F((a, b)) = \oplus_{i=1}^{2n-a-b} E_i(\mathrm{Hom}(T_i[-1], (a, b)))$ (as vector spaces). When $1 \le i \le n-a+1$, there are non-zero $\mathcal{T}-$maps and nonzero $\mathcal{D}-$maps from $T_i[-1]$ to (a, b) and when $n+2-a \le i \le 2n-a-b$, there are not non-zero $\mathcal{D}-$maps from $T_i[-1]$ to (a, b). So there is a basis $\{E_1(f_1), \cdots, E_{n-a+1}(f_{n-a+1}), E_1(g_1), \cdots, E_{2n-a-b}(g_{2n-a-b})\}$ of $F((a, b))$ where f_i (g_i) is a nonzero $\mathcal{T}-$map ($\mathcal{D}-$map respectively) in $\mathrm{Hom}(T_i[-1], (a, b))$. Actually f_1 is a right $\mathrm{add}T-$approximation of $(a, n-1)$ due to lemma 3.1 and it proof. It follows that $E_1(f_1)B = F((a, b))$. There are no nonzero $\mathcal{D}-$maps from $T_j[-1]$ to $T_i[-1]$ for any $i \ge 2$. Then $E_i(f_i)B = \oplus_{j=i}^{n-a+1} E_j(f_j)k$ $(2 \le i \le n-a+1)$, Since any $\mathcal{D}-$map from $T_j[-1]$ to $(a, n-1)$ factors the ray starting at $T_j[-1]$ [BMV], then $E_i(g_i)B = \oplus_{j=i}^{2n-a-b} E_j(g_j)$

$(1 \leq i \leq 2n-a-b)$. Now we determine the $B-$submodules $F(Y)$ of $F((a,b))$ with $\dim\mathrm{Hom}(T_1[-1],Y) \neq 1$, and their Euler characteristic of the Grassmann associated to the dimension vectors of such $F(Y)$.

If $f_1 \in F(Y)$, then $F(Y) = F((a,b))$, $e = \underline{\dim}(FY) = \sum_{k=1}^{n-a+1} 2e_k + \sum_{k=n-a+2}^{2n-a-b} e_k$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = \mathrm{ind}_T(a,b) + \mathrm{ind}_T(a-1,b) = ([T_1]-[T_{n-a+1}]) - [T_{2n-a-b}]) + ([T_1] - [T_{n-a+2}]) - [T_{2n-a-b+1}])$, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1],(a,n-1)))) = 1$.

Now we assume $f_1 \notin F(Y)$. In this case, we have $g_1 \notin F(Y)$ otherwise $\dim\mathrm{Hom}(T_1[-1],Y) = 1$.

We consider the sets:

$$M_t = \{i \mid 1 \leq i \leq n-a+1, \ \ \dim(FY \cap \mathrm{Hom}(T_i[-1],(a,n-1))) = t\}, \ \ t = 1,2,$$

and

$$M_3 = \{i \mid n-a+2 \leq i \leq 2n-a-b, \ \ \dim(FY \cap \mathrm{Hom}(T_i[-1],(a,n-1))) = 1\}.$$

We assume j_0 (i_0, k_0) is the minimal number in M_1 (M_2, M_3 respectively) if the corresponding set is nonempty.

There are seven subcases. When $b = n-1$, the subcases (b), (e), (f) do not exist, the subcase (d) is different from the situation that $b < n-1$ and in the subcase (g), both the two submodules in (i) and (ii) are isomorphic. We recall that $E_i : \mathrm{Hom}(T_i[-1],Y) \to \mathrm{Hom}(T[-1],Y)$ denotes the natural embedding.

(a) $M_t = \emptyset$ for $t = 1,2,3$. Then $e = \underline{\dim}(FY) = 0$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = 0$, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1],(a,n-1)))) = 1$.

(b) $M_t = \emptyset$ for $t = 1,2$ and $M_3 \neq \emptyset$. Then $E_{k_0}(g_{k_0}) \in FY$ and $FY \cong E_{k_0}(g_{k_0})B$ which is indecomposable. Therefore $e = \underline{\dim}(FY) = \sum_{k=k_0}^{2n-a-b} e_k$ and $Y \cong (a+b-n+1, 2n-a-b-k_0+1)$, $n-a+2 \leq k_0 \leq 2n-a-b$. In this case, $\mathrm{ind}_T FY + \mathrm{ind}_T FY[1] = [T_{k_0-1}] - [T_{2n-a-b}] + [T_{k_0}] - [T_{2n-a-b+1}]$, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1],(a,n-1)))) = 1$.

(c) $M_1 = \emptyset$ and $M_2 \neq \emptyset$. Then $i_0 \geq 2$ otherwise $i_0 = 1$ which implies $E_1(f_1) \in F(Y)$ contradicting to our assumption. Then $E_{i_0}(f_{i_0}), E_{i_0}(g_{i_0}) \in F(Y)$, and $F(Y) \cong E_{i_0}(f_{i_0})B \oplus E_{i_0}(g_{i_0})B$. Then $e = \underline{\dim}(FY) = \sum_{k=i_0}^{n-a+1} 2e_k + \sum_{k=n-a+2}^{2n-a-b} e_k$, $2 \leq i_0 \leq n-a+1$. From the construction of $E_{i_0}(f_{i_0})B$ and $E_{i_0}(g_{i_0})B$, they are indecomposable with dimension vectors $\sum_{k=i_0}^{n-a+1} e_k$ and $\sum_{k=i_0}^{2n-a-b} e_k$ respectively. Then $E_{i_0}(f_{i_0})B \cong F((a_1,b_1))$ and $E_{i_0}(g_{i_0})B \cong F((a_2,b_2))$, $(a_s,b_s) \in \mathcal{D}$, for $s = 1,2$. Comparing the dimension vectors of $F(X)$, where X is indecomposable in $\mathcal{D}$, we have that $(a_s,b_s) \in I$. Therefore $(a_1,b_1) = (a, n-a-i_0+2)$, $(a_2,b_2) = (a+b-n+1, 2n-a-b-i_0+1)$ and then $Y \cong (a,n-a-i_0+2) \oplus (a+b-n+1, 2n-a-b-i_0+1)$, $2 \leq i_0 \leq n-a+1$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{i_0-1}] - [T_{n-a+1}] + [T_{i_0-1}] - [T_{2n-a-b}] + [T_{i_0}] - [T_{n-a+2}] + [T_{i_0}] - [T_{2n-a-b+1}] = 2[T_{i_0-1}] - [T_{n-a+1}] - [T_{2n-a-b}] + 2[T_{i_0}] - [T_{n-a+2}] - [T_{2n-a-b+1}]$, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1],(a,b)))) = 1$.

(d) $M_1 \neq \emptyset$, $M_2 = \emptyset$ and $M_3 = \emptyset$. Then $j_0 \geq 2$ otherwise $j_0 = 1$ which contradicts to the fact that $\dim\mathrm{Hom}(T[-1],Y) \neq 1$. If $b \neq n-1$, then $E(f_{j_0}) \in F(Y)$ otherwise we have that $F(Y)$ contains an non-zero element $E_{j_0}(f_{j_0} + \lambda g_{j_0})$ with $\lambda \neq 0$, which implies

$M_3 \neq \emptyset$, a contradiction. Then $F(Y) \cong E_{j_0}(f_{j_0})B$. Similar as above, we have $F(Y)$ is indecomposable with dimension vector $\sum_{k=j_0}^{n-a+1} e_k$. Then $F(Y) \cong F((a, n-a-j_0+2))$. Therefore $Y \cong (a, n-a-j_0+2), 2 \leq j_0 \leq n-a+1$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{j_0-1}] - [T_{n-a+1}] + [T_{j_0}] - [T_{n-a+2}], \chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a, b)))) = 1$.

If $b = n-1$, then for any non-zero map $h_{j_0} \in \mathrm{Hom}(T_{j_0}, (a, b))$, $E_{j_0}(h_{j_0}) \in F(Y)$ and $F(Y) \cong E_{j_0}(h_{j_0})B$. As the same reason, $F(Y)$ is indecomposable with dimension vector $\sum_{k=j_0}^{n-a+1} e_k$. Therefore $Y \cong (a, n-a-j_0+2), 2 \leq j_0 \leq n-a+1$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{j_0-1}] - [T_{n-a+1}] + [T_{j_0}] - [T_{n-a+2}], \chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a, b)))) = 2$.

(e) $M_1 \neq \emptyset$, $M_2 = \emptyset$, $M_3 \neq \emptyset$ and $k_0 = n-a+2$. Then $j_0 \geq 2$. There is a nonzero map $h_{j_0} \in \mathrm{Hom}(T_{j_0}, (a, b))$ such that $E_{j_0}(h_{j_0}) \in F(Y)$. There are two different cases:

(i) h_{j_0} is a scalar of f_{j_0}. Then $E_{n-a+2}(g_{n-a+2}) \in F(Y)$, but $E_{n-a+2}(g_{n-a+2}) \notin E_{j_0}(f_{j_0)})B$. We have that $FY \cong E_{j_0}(f_{j_0)})B \oplus E_{n-a+2}(g_{n-a+2})B$, $E_{j_0}(f_{j_0)})B$ and $E_{n-a+2}(g_{n-a+2})B$ are indecomposable with dimension vector $\sum_{k=j_0}^{n-a+1} e_k$ and $\sum_{k=n-a+2}^{2n-a-b} e_k$ respectively. Then $F(Y) \cong F((a, n-a-j_0+2)) \oplus F((a+b-n+1, n-b-1))$ with $\underline{\dim} FY = \sum_{k=j_0}^{2n-a-b} e_k$. Therefore $Y \cong (a, n-a-j_0+2) \oplus (a+b-n+1, n-b-1)$, $2 \leq j_0 \leq n-a+1$. In this case, we have

$$
\begin{aligned}
& \mathrm{ind}_T((a, n-a-i_0+2) \oplus (a+b-n+1, n-b-1)) + \\
& \mathrm{ind}_T((a, n-a-i_0+2) \oplus (a+b-n+1, n-b-1))[1] \\
= \; & [T_{j_0-1}] - [T_{n-a+1}] + [T_{n-a+1}] - [T_{2n-a-b}] + \\
& [T_{j_0}] - [T_{n-a+2}] + [T_{n-a+2}] - [T_{2n-a-b+1}] \\
= \; & [T_{j_0-1}] - [T_{2n-a-b}] + [T_{j_0}] - [T_{2n-a-b+1}]
\end{aligned}
$$

(ii) h_{j_0} is not a scalar of f_{j_0}. Then $FY \cong E_{j_0}(h_{j_0})B$, which is indecomposable with dimension vector $\sum_{k=j_0}^{2n-a-b} e_k$. Then $F(Y) \cong F((a+b-n+1, 2n-a-b-j_0+1))$. Therefore $Y \cong (a+b-n+1, 2n-a-b-j_0+1)$, $2 \leq j_0 \leq n-a+1$. We have that

$$
\begin{aligned}
& \mathrm{ind}_T(a+b-n+1, 2n-a-b-j_0+1) + \\
& \mathrm{ind}_T(a+b-n+1, 2n-a-b-j_0+1)[1] \\
= \; & [T_{j_0-1}] - [T_{2n-a-b}] + [T_{j_0}] - [T_{2n-a-b+1}].
\end{aligned}
$$

In all case, the indexes $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1]$ are equal to each other, and the dimension vector of $F(Y)$ is $e = \sum_{k=j_0}^{2n-a-b} e_k$. Therefore, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{j_0-1}] - [T_{2n-a-b}] + [T_{j_0}] - [T_{2n-a-b+1}], \chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a, b)))) = 2$.

(f) $M_1 \neq \emptyset$, $M_2 = \emptyset$, $M_3 \neq \emptyset$ and $k_0 > n-a+2$. Then $j_0 \geq 2$. Then we have that $E_{j_0}(f_{j_0)})B, \in F(Y)$ otherwise similar as the case (e), we will have $k_0 = n-a+2$, a contradiction. So we have that $E_{j_0}(f_{j_0)})B, E_{n-a+2}(g_(n-a+2)) \in F(Y)$ and then $FY \cong E_{j_0}(f_{j_0)})B \oplus E_{n-a+2}(g_(n-a+2))B$, $E_{j_0}(f_{j_0)})B$ and $E_{n-a+2}(g_(n-a+2))B$ are indecomposable with dimension vector $\sum_{k=j_0}^{n-a+1} e_k$ and $\sum_{k=k_0}^{2n-a-b} e_k$ respectively. Then $F(Y) \cong F((a, n-a-j_0+2)) \oplus F((a+b-n+1, 2n-a-b-k_0+1))$ with $e = \underline{\dim} FY = \sum_{k=j_0}^{n-a+1} e_k + \sum_{k=k_0}^{2n-a-b} e_k$. Therefore $Y \cong (a, n-a-j_0+2) \oplus (a+b-n+1, 2n-a-b-k_0+1)$, $n-a+3 \leq k_0 \leq 2n-a-b$, $2 \leq j_0 \leq n-a+1$. In this case, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{j_0-1}] - [T_{n-a+1}] + [T_{k_0-1}] - [T_{2n-a-b}] + [T_{k_0}] - [T_{2n-a-b+1}] + [T_{j_0}] - [T_{n-a+2}], \chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a, b)))) = 1$.

(g) $M_1 \neq \emptyset$ and $M_2 \neq \emptyset$. Then $i_0 > j_0 \geq 2$. There is a non-zero map $h_{j_0} \in \mathrm{Hom}(T_{j_0}, (a,b))$ such that $E_{j_0}(h_{j_0}), E_{i_0}(f_{i_0}), E_{i_0}(g_{i_0}) \in F(Y)$.

(i) h_{j_0} is a scalar of f_{j_0}. Taking a non-zero map $h_{i_0} : T_{i_0} \to (a,b)$ which belongs to $F(Y)$ but not to $E_{j_0}(h_{j_0})B$, we have that $F(Y) \cong E_{j_0}(h_{j_0})B \oplus E_{i_0}(h_{i_0})B$, each of them is indecomposable. $\underline{\dim}(E_{j_0}(h_{j_0})B) = \sum_{k=j_0}^{n-a+1} e_k$, $\underline{\dim}(E_{i_0}(h_{i_0})B) = \sum_{k=i_0}^{2n-a-b} e_k$. Then $e = \underline{\dim}(FY) = \sum_{k=j_0}^{n-a+1} e_k + \sum_{k=i_0}^{2n-a-b} e_k$, and $F(Y) \cong F((a, n-a-j_0+2)) \oplus F((a+b-n+1, 2n-a-b-i_0+1))$. Therefore, $Y \cong (a, n-a-j_0+2) \oplus (a+b-n+1, 2n-a-b-i_0+1)$, $2 \leq j_0 < i_0 \leq n-a+1$. In this case, we have that

$$\begin{aligned}
& \mathrm{ind}_T((a, n-a-j_0+2) \oplus (a+b-n+1, 2n-a-b-i_0+1)) + \\
& \mathrm{ind}_T((a, n-a-j_0+2) \oplus (a+b-n+1, 2n-a-b-i_0+1))[1] \\
= \; & [T_{j_0-1}] - [T_{n-a+1}] + [T_{i_0-1}] - [T_{2n-a-b}] + \\
& [T_{j_0}] - [T_{n-a+2}] + [T_{i_0}] - [T_{2n-a-b+1}]
\end{aligned}$$

(ii) h_{j_0} is not a scalar of f_{j_0}. Then $E_{i_0}(f_{i_0}) \in FY$ and $FY \cong E_{j_0}(h_{j_0})B \oplus E_{i_0}(f_{i_0})B$, each of them is indecomposable. $\underline{\dim}(E_{j_0}(h_{j_0})B) = \sum_{k=j_0}^{2n-a-b} e_k$, $\underline{\dim}(E_{i_0}(f_{i_0})B) = \sum_{k=i_0}^{n-a+1} e_k$. Then $e = \underline{\dim}(FY) = \sum_{k=j_0}^{2n-a-b} e_k + \sum_{k=i_0}^{n-a+1} e_k$, and $F(Y) \cong F((a+b-n+1, 2n-a-b-j_0+1)) \oplus F((a, n-a-i_0+2))$. Therefore, $Y \cong (a+b-n+1, 2n-a-b-j_0+1) \oplus (a, n-a-i_0+2)$, $2 \leq j_0 < i_0 \leq n-a+1$. We have that

$$\begin{aligned}
& \mathrm{ind}_T((a+b-n+1, 2n-a-b-j_0+1) \oplus (a, n-a-i_0+2)) + \\
& \mathrm{ind}_T((a+b-n+1, 2n-a-b-j_0+1) \oplus (a, n-a-i_0+2))[1] \\
= \; & [T_{j_0-1}] - [T_{2n-a-b}] + [T_{i_0-1}] - [T_{n-a+1}] + \\
& [T_{j_0}] - [T_{2n-a-b+1}] + [T_{i_0}] - [T_{n-a+2}]
\end{aligned}$$

The indexes $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1]$ in the two subcases are equal to each other, and the dimension vector of $F(Y)$ is $e = \sum_{k=j_0}^{2n-a-b} e_k + \sum_{k=i_0}^{n-a+1} e_k$. Therefore, $\mathrm{ind}_T Y + \mathrm{ind}_T Y[1] = [T_{j_0}] - [T_{n-a+2}] + [T_{i_0}] - [T_{2n-a-b+1}]$, $\chi(\mathrm{Gr}_e(\mathrm{Hom}(T[-1], (a,b)))) = 2$.

We know that $\mathrm{ind}_T(a,b) = [T_1] - [T_{n-a+1}] - [T_{2n-a-b}]$. When $b = n-1$, by Definition 3.3, we have that

$$\begin{aligned}
X_{(a,n-1)} = \; & \frac{x_1}{x_{n-a+1}x_{2n-a-(n-1)}}\Big(\frac{x_{n-a+1}x_{2n-a-(n-1)}x_{n-a+2}x_{2n-a-(n-1)+1}}{x_1^2} + 1 + \sum_{k=2}^{n-a+1} \frac{x_{n-a+1}x_{2n-a-(n-1)}x_{n-a+2}x_{2n-a-(n-1)+1}}{x_{k-1}^2 x_k^2} \\
& +2\sum_{k=2}^{n-a+1} \frac{x_{n-a+1}x_{n-a+2}}{x_{k-1}x_k} + 2\sum_{2\leq k<l\leq n-a+1} \frac{x_{n-a+1}x_{2n-a-(n-1)}x_{n-a+2}x_{2n-a-(n-1)+1}}{x_{k-1}x_k x_{l-1}x_l}\Big) \\
= \; & x_1 x_{n-a+2}x_{2n-a-(n-1)+1}\Big(\frac{1}{x_1^2} + \frac{1}{x_{n-a+1}x_{n-a+2}}\frac{1}{x_{2n-a-(n-1)}x_{2n-a-(n-1)+1}} + \sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}^2 x_k^2} \\
& +2\frac{1}{x_{2n-a-(n-1)}x_{2n-a-(n-1)+1}}\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k} + 2\sum_{2\leq k<l\leq n-a+1}\frac{1}{x_{k-1}x_k x_{l-1}x_l}\Big) \\
= \; & x_1 x_{n-a+2}x_{2n-a-(n-1)+1}\Big(\frac{1}{x_1^2} + (\frac{1}{x_{n-a+1}x_{n-a+2}})^2 + 2\frac{1}{x_{n-a+1}x_{n-a+2}}\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k} \\
& (\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k})^2\Big) \\
= \; & x_1 x_{n-a+2}x_{2n-a-(n-1)+1}\left(\frac{1}{x_1^2} + (\sum_{k=2}^{n-a+2}\frac{1}{x_{k-1}x_k})^2\right) \\
= \; & x_1 x_{n-a+2}x_{2n-a-(n-1)+1}\left(\frac{1}{x_1^2} + \left(\sum_{k=2}^{n-a+2}\frac{1}{x_{k-1}x_k}\right)\left(\sum_{l=2}^{2n-a-(n-1)+1}\frac{1}{x_{l-1}x_l}\right)\right) \\
= \; & x_1 x_{n-a+2}x_{2n-a-(n-1)+1}\left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1}\frac{1}{x_k x_{k+1}}\right)\left(\sum_{l=1}^{2n-a-(n-1)}\frac{1}{x_l x_{l+1}}\right)\right).
\end{aligned}$$

When $b < n-1$, by Definition 3.3, we have that

$$
\begin{aligned}
X_{(a,b)} &= \frac{x_1}{x_{n-a+1}x_{2n-a-b}}\Big(\frac{x_{n-a+1}x_{2n-a-b}x_{n-a+2}x_{2n-a-b+1}}{x_1^2} + 1 + \sum_{k=n-a+2}^{2n-a-b}\frac{x_{2n-a-b}x_{2n-a-b+1}}{x_{k-1}x_k}\\
&\quad + \sum_{k=2}^{n-a+1}\frac{x_{n-a+1}x_{2n-a-b}x_{n-a+2}x_{2n-a-b+1}}{x_{k-1}^2x_k^2} + \sum_{k=2}^{n-a+1}\frac{x_{n-a+1}x_{n-a+2}}{x_{k-1}x_k} + 2\sum_{k=2}^{n-a+1}\frac{x_{2n-a-b}x_{2n-a-b+1}}{x_{k-1}x_k}\\
&\quad + \sum_{l=n-a+3}^{2n-a-b}\sum_{k=2}^{n-a+1}\frac{x_{n-a+1}x_{2n-a-b}x_{n-a+2}x_{2n-a-b+1}}{x_{k-1}x_kx_{l-1}x_l} + 2\sum_{2\le k<l\le n-a+1}\frac{x_{n-a+1}x_{2n-a-b}x_{n-a+2}x_{2n-a-b+1}}{x_{k-1}x_kx_{l-1}x_l}\Big)\\
&= x_1x_{n-a+2}x_{2n-a-b+1}\Big(\frac{1}{x_1^2} + \frac{1}{x_{n-a+1}x_{n-a+2}}\frac{1}{x_{2n-a-b}x_{2n-a-b+1}} + \frac{1}{x_{n-a+1}x_{n-a+2}}\sum_{k=n-a+2}^{2n-a-b}\frac{1}{x_{k-1}x_k}\\
&\quad + \sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}^2x_k^2} + \frac{1}{x_{2n-a-b}x_{2n-a-b+1}}\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k} + 2\frac{1}{x_{n-a+1}x_{n-a+2}}\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k}\\
&\quad + \sum_{l=n-a+3}^{2n-a-b}\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_kx_{l-1}x_l} + 2\sum_{2\le k<l\le n-a+1}\frac{1}{x_{k-1}x_kx_{l-1}x_l}\Big)\\
&= x_1x_{n-a+2}x_{2n-a-b+1}\Big(\frac{1}{x_1^2} + \frac{1}{x_{n-a+1}x_{n-a+2}}\big(\frac{1}{x_{2n-a-b}x_{2n-a-b+1}} + \sum_{k=n-a+2}^{2n-a-b}\frac{1}{x_{k-1}x_k} + \sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k}\big)\\
&\quad +\big(\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k}\big)\big(\frac{1}{x_{2n-a-b}x_{2n-a-b+1}} + \frac{1}{x_{n-a+1}x_{n-a+2}} + \sum_{l=n-a+3}^{2n-a-b}\frac{1}{x_{l-1}x_l}\big)\\
&\quad +\big(\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}^2x_k^2} + 2\sum_{2\le k<l\le n-a+1}\frac{1}{x_{k-1}x_kx_{l-1}x_l}\big)\Big)\\
&= x_1x_{n-a+2}x_{2n-a-b+1}\Big(\frac{1}{x_1^2} + \frac{1}{x_{n-a+1}x_{n-a+2}}\sum_{k=2}^{2n-a-b+1}\frac{1}{x_{k-1}x_k}\\
&\quad +\big(\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k}\big)\big(\sum_{l=n-a+2}^{2n-a-b+1}\frac{1}{x_{l-1}x_l}\big)\\
&\quad +\big(\sum_{k=2}^{n-a+1}\frac{1}{x_{k-1}x_k}\big)\big(\sum_{l=2}^{n-a+1}\frac{1}{x_{l-1}x_l}\big)\Big)\\
&= x_1x_{n-a+2}x_{2n-a-b+1}\left(\frac{1}{x_1^2} + \Big(\sum_{k=2}^{n-a+2}\frac{1}{x_{k-1}x_k}\Big)\Big(\sum_{l=2}^{2n-a-b+1}\frac{1}{x_{l-1}x_l}\Big)\right)\\
&= x_1x_{n-a+2}x_{2n-a-b+1}\left(\frac{1}{x_1^2} + \Big(\sum_{k=1}^{n-a+1}\frac{1}{x_kx_{k+1}}\Big)\Big(\sum_{l=1}^{2n-a-b}\frac{1}{x_lx_{l+1}}\Big)\right).
\end{aligned}
$$

Thus the proof is completed. □

We give an example to illustrate the result above:

Example 1. *In Γ_5, there are twenty indecomposable rigid objects:* (1, 1)*,* (1, 2)*,* (1, 3)*,* (1, 4)*,* (2, 1)*,* (2, 2)*,* (2, 3)*,* (2, 4)*,* (3, 1)*,* (3, 2)*,* (3, 3)*,* (3, 4)*,* (4, 1)*,* (4, 2)*,* (4, 3)*,* (4, 4)*,* (5, 1)*,* (5, 2)*,* (5, 3)*,* (5, 4) *(see Fig.2). They belong to three disjoint subsets:* $O = \{(1,1),(1,2),(1,3),(1,4)\}$*,* $I = \{(2,1),(2,2),(2,3),(3,1),(3,2),(4,1)\}$*,* $II = \{(2,4),(3,4),(4,4),(5,4),(3,3),(4,3),(5,3),(4,2),(5,2),(5,1)\}$.

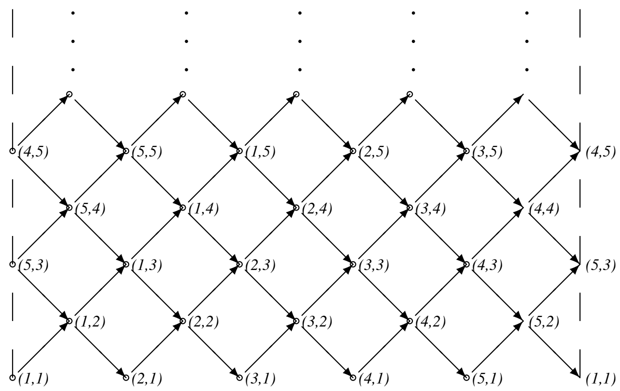

Fig.2 The tube of rank 5.

By Theorem 3.7, we have the following result where we reserve the details of the calculation where the submodules whose first items of dimension vectors are 1 are omitted:

subsets	(a,b)	*submodule FY*	$e = \underline{dim}FY$	$x^{-(ind_T Y+ind_T Y[1])}$	χ	$x^{ind_T(a,b)}$	$X_{(a,b)}$
O	$(1,1)$	*0*	*(0,0,0,0)*	1	*1*	x_4	x_4
	$(1,2)$	*0*	*(0,0,0,0)*	1	*1*	x_3	x_3
	$(1,3)$	*0*	*(0,0,0,0)*	1	*1*	x_2	x_2
	$(1,4)$	*0*	*(0,0,0,0)*	1	*1*	x_1	x_1
I	$(2,1)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_3}{x_4}$	$x_3\left(\frac{1}{x_4}+\frac{1}{x_3x_4}\right)$
		F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$(2,2)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_2}{x_4}$	$x_2\left(\frac{1}{x_4}+\frac{1}{x_3x_4}+\frac{1}{x_2x_3}\right)$
		F(2,2)	*(0,0,1,1)*	$\frac{x_4}{x_2}\frac{1}{x_3}$	*1*		
		F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$(2,3)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_4}$	$x_1\left(\frac{1}{x_4}+\frac{1}{x_3x_4}+\frac{1}{x_2x_3}+\frac{1}{x_1x_2}\right)$
		F(2,3)	*(0,1,1,1)*	$\frac{x_4}{x_1}\frac{1}{x_2}$	*1*		
		F(2,2)	*(0,0,1,1)*	$\frac{x_4}{x_2}\frac{1}{x_3}$	*1*		
		F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$(3,1)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_2}{x_3}$	$x_4x_2\left(\frac{1}{x_3x_4}+\frac{1}{x_2x_3}\right)$
		F(3,1)	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*1*		
	$(3,2)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_2}{x_4}$	$x_4x_2\left(\frac{1}{x_3x_4}+\frac{1}{x_2x_3}+\frac{1}{x_1x_2}\right)$
		F(3,2)	*(0,1,1,0)*	$\frac{x_3}{x_1}\frac{x_4}{x_2}$	*1*		
		F(3,1)	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*1*		
	$(4,1)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_2}$	$x_3x_1\left(\frac{1}{x_2x_3}+\frac{1}{x_1x_2}\right)$
		F(4,1)	*(0,1,0,0)*	$\frac{x_2}{x_1}\frac{x_3}{x_2}$	*1*		
II	$(2,4)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_4^2}$	$\frac{x_1}{x_4^2}\Big(1+\frac{x_4^2}{x_1^2}+\sum_{k=1}^{3}\frac{x_4^2}{x_k^2x_{k+1}^2}$ $+2\sum_{k=1}^{3}\frac{x_4}{x_kx_{k+1}}$ $+2\sum_{1\leq k<l\leq 3}\frac{x_4^2}{x_kx_lx_{k+1}x_{l+1}}\Big)$ $=x_1\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{4}\frac{1}{x_kx_{k+1}}\right)^2\right)$
		F(2,4)	*(2,2,2,2)*	$\frac{x_4^2}{x_1}\frac{1}{x_1}$	*1*		
		$F((2,3)\oplus(2,3))$	*(0,2,2,2)*	$\frac{x_4^2}{x_1^2}\frac{1}{x_2^2}$	*1*		
		$F((2,2)\oplus(2,2))$	*(0,0,2,2)*	$\frac{x_4^2}{x_2^2}\frac{1}{x_3^2}$	*1*		
		$F((2,1)\oplus(2,1))$	*(0,0,0,2)*	$\frac{x_4^2}{x_3^2}\frac{1}{x_4^2}$	*1*		
		F(2,3)	*(0,1,1,1)*	$\frac{x_4}{x_1}\frac{1}{x_2}$	*2*		
		F(2,2)	*(0,0,1,1)*	$\frac{x_4}{x_2}\frac{1}{x_3}$	*2*		
		F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*2*		
		$F((2,3)\oplus(2,2))$	*(0,1,2,2)*	$\frac{x_4^2}{x_1x_2}\frac{1}{x_2x_3}$	*2*		
		$F((2,3)\oplus(2,1))$	*(0,1,1,2)*	$\frac{x_4^2}{x_1x_3}\frac{1}{x_2x_4}$	*2*		
		$F((2,2\oplus(2,1))$	*(0,0,1,2)*	$\frac{x_4^2}{x_2x_3}\frac{1}{x_3x_4}$	*2*		
	$(3,4)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_3^2}$	$\frac{x_1}{x_3^2}\Big(1+\frac{x_3^2x_4^2}{x_1^2}+\sum_{k=1}^{2}\frac{x_3^2x_4^2}{x_k^2x_{k+1}^2}$ $+2\sum_{k=1}^{2}\frac{x_3x_4}{x_kx_{k+1}}+2\frac{x_3^2x_4^2}{x_1x_2^2x_3}\Big)$ $=x_1x_4^2\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{3}\frac{1}{x_kx_{k+1}}\right)^2\right)$
		F(3,4)	*(2,2,2,0)*	$\frac{x_3^2}{x_1}\frac{x_4^2}{x_1}$	*1*		
		$F((3,2)\oplus(3,2))$	*(0,2,2,0)*	$\frac{x_3^2}{x_1^2}\frac{x_4^2}{x_2^2}$	*1*		
		$F((3,1)\oplus(3,1))$	*(0,0,2,0)*	$\frac{x_3^2}{x_2^2}\frac{x_4^2}{x_3^2}$	*1*		
		F(3,2)	*(0,1,1,0)*	$\frac{x_3}{x_1}\frac{x_4}{x_2}$	*2*		
		F(3,1)	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*2*		

	$F((3,2)\oplus(3,1))$	*(0,1,2,0)*	$\frac{x_3^2}{x_1x_2}\frac{x_4^2}{x_2x_3}$	*2*		
$(4,4)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_2^2}$	$\frac{x_1}{x_2^2}\left(1+\frac{x_2^2x_3^2}{x_1^2}+\frac{x_2^2x_3^2}{x_1^2x_2^2}+2\frac{x_2x_3}{x_1x_2}\right)$ $=x_1x_3^2\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{2}\frac{1}{x_kx_{k+1}}\right)^2\right)$
	F(4,4)	*(2,2,0,0)*	$\frac{x_2^2}{x_1}\frac{x_3^2}{x_1}$	*1*		
	$F((4,1)\oplus(4,1))$	*(0,2,0,0)*	$\frac{x_2^2}{x_1^2}\frac{x_3^2}{x_2^2}$	*1*		
	F(4,1)	*(0,1,0,0)*	$\frac{x_2}{x_1}\frac{x_3}{x_2}$	*2*		
$(5,4)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_1^2}$	$\frac{x_1}{x_1^2}\left(1+\frac{x_1^2x_2^2}{x_1^2}\right)$ $=x_1x_2^2\left(\frac{1}{x_1^2}+\left(\frac{1}{x_1x_2}\right)^2\right)$
	F(5,4)	*(2,0,0,0)*	$\frac{x_1^2}{x_1}\frac{x_2^2}{x_1}$	*1*		
$(3,3)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_3x_4}$	$\frac{x_1}{x_3x_4}\left(1+\frac{x_3x_4^2}{x_1^2}+\frac{x_4}{x_3x_4}+\frac{x_3x_4^2}{x_1^2x_2^2}+\frac{x_3x_4^2}{x_2^2x_3^2}+\sum_{k=1}^{2}\frac{x_3x_4}{x_kx_{k+1}}+2\sum_{k=1}^{2}\frac{x_4}{x_kx_{k+1}}+2\frac{x_3x_4^2}{x_1x_2^2x_3}\right)$ $=x_1x_4\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{3}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{k=1}^{4}\frac{1}{x_kx_{k+1}}\right)\right)$
	F(3,3)	*(2,2,2,1)*	$\frac{x_3x_4}{x_1}\frac{x_4}{x_1}$	*1*		
	F(2,1)	*(2,2,2,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$F((3,2)\oplus(2,3))$	*(0,2,2,1)*	$\frac{x_3x_4}{x_1^2}\frac{x_4}{x_2^2}$	*1*		
	$F((3,1)\oplus(2,2))$	*(0,0,2,1)*	$\frac{x_3x_4}{x_2^2}\frac{x_4}{x_3^2}$	*1*		
	F(3,2)	*(0,1,1,0)*	$\frac{x_3}{x_1}\frac{x_4}{x_2}$	*1*		
	F(3,1)	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*1*		
	$F((3,2)\oplus(2,1))$	*(0,1,1,1)*	$\frac{x_3x_4}{x_1x_3}\frac{x_4}{x_2x_4}$ $=\frac{x_4}{x_1}\frac{1}{x_2}$	*2*		
	$F(2,3)$	*(0,1,1,1)*				
	$F((3,1)\oplus(2,1))$	*(0,0,1,1)*	$\frac{x_3x_4}{x_2x_3}\frac{x_4}{x_3x_4}$ $=\frac{x_4}{x_2}\frac{1}{x_3}$	*2*		
	$F(2,2)$	*(0,0,1,1)*				
	$F((3,2)\oplus(2,2))$	*(0,1,2,1)*	$\frac{x_3x_4}{x_1x_2}\frac{x_4}{x_2x_3}$ $=\frac{x_4x_3}{x_1x_2}\frac{x_4}{x_2x_3}$	*2*		
	$F((2,3)\oplus(3,1))$	*(0,1,2,1)*				
$(4,2)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_2x_4}$	$\frac{x_1}{x_2x_4}\left(1+\frac{x_2x_3x_4}{x_1^2}+\frac{x_4}{x_2x_3}+\frac{x_4}{x_3x_4}+\frac{x_2x_3x_4}{x_1^2x_2^2}+\frac{x_2x_3}{x_1x_2}+2\frac{x_4}{x_1x_2}+\frac{x_2x_3x_4}{x_1x_2x_3x_4}\right)$ $=x_1x_3\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{2}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{k=1}^{4}\frac{1}{x_kx_{k+1}}\right)\right)$
	F(4,2)	*(2,2,1,1)*	$\frac{x_2x_4}{x_1}\frac{x_3}{x_1}$	*1*		
	F(2,2)	*(0,0,1,1)*	$\frac{x_4}{x_2}\frac{1}{x_3}$	*1*		
	F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$F((4,1)\oplus(2,3))$	*(0,2,1,1)*	$\frac{x_2x_4}{x_1^2}\frac{x_3}{x_2^2}$	*1*		
	F(4,1)	*(0,1,0,0)*	$\frac{x_2}{x_1}\frac{x_3}{x_2}$	*1*		
	$F((4,1)\oplus(2,2))$	*(0,1,1,1)*	$\frac{x_2x_4}{x_1x_2}\frac{x_3}{x_2x_3}$ $=\frac{x_4}{x_1}\frac{1}{x_2}$	*2*		
	$F(2,3)$	*(0,1,1,1)*				
	$F((4,1)\oplus(2,1))$	*(0,1,0,1)*	$\frac{x_2x_4}{x_1x_3}\frac{x_3}{x_2x_4}$	*1*		
$(4,3)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_2x_3}$	$\frac{x_1}{x_2x_3}\left(1+\frac{x_2x_3^2x_4}{x_1^2}+\frac{x_3x_4}{x_2x_3}+\frac{x_2x_3^2x_4}{x_1^2x_2^2}+\frac{x_2x_3}{x_1x_2}+2\frac{x_3x_4}{x_1x_2}\right)$ $=x_1x_3x_4\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{2}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{k=1}^{3}\frac{1}{x_kx_{k+1}}\right)\right)$
	F(4,3)	*(2,2,1,1)*	$\frac{x_2x_3}{x_1}\frac{x_3x_4}{x_1}$	*1*		
	F(3,1)	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*1*		
	$F((4,1)\oplus(3,2))$	*(0,2,1,0)*	$\frac{x_2x_3}{x_1^2}\frac{x_3x_4}{x_2^2}$	*1*		
	F(4,1)	*(0,1,0,0)*	$\frac{x_2}{x_1}\frac{x_3}{x_2}$	*1*		
	$F((4,1)\oplus(3,1))$	*(0,1,1,0)*	$\frac{x_2x_3}{x_1x_2}\frac{x_3x_4}{x_2x_3}$ $=\frac{x_3}{x_1}\frac{x_4}{x_2}$	*2*		
	F(3,2)	*(0,1,1,0)*				
$(5,1)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_1x_4}$	$\frac{x_1}{x_1x_4}\left(1+\frac{x_1x_2x_4}{x_1^2}+\sum_{k=1}^{3}\frac{x_4}{x_kx_{k+1}}\right)$ $=x_1x_2\left(\frac{1}{x_1^2}+\frac{1}{x_1x_2}\left(\sum_{k=1}^{4}\frac{1}{x_kx_{k+1}}\right)\right)$
	F(5,1)	*(2,1,1,1)*	$\frac{x_1x_4}{x_1}\frac{x_2}{x_1}$	*1*		
	F(2,3)	*(0,1,1,1)*	$\frac{x_4}{x_1}\frac{1}{x_2}$	*1*		
	$F(2,2)$	*(0,0,1,1)*	$\frac{x_4}{x_2}\frac{1}{x_3}$	*1*		

		F(2,1)	*(0,0,0,1)*	$\frac{x_4}{x_3}\frac{1}{x_4}$	*1*		
	$(5,2)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_1x_3}$	$\frac{x_1}{x_1x_3}\left(1+\frac{x_1x_2x_3x_4}{x_1^2}+\sum_{k=1}^{2}\frac{x_3x_4}{x_kx_{k+1}}\right)$ $=x_1x_2x_4\left(\frac{1}{x_1^2}+\frac{1}{x_1x_2}\left(\sum_{k=1}^{3}\frac{1}{x_kx_{k+1}}\right)\right)$
		F(5,2)	*(2,1,1,0)*	$\frac{x_1x_3}{x_1}\frac{x_2x_4}{x_1}$	*1*		
		F(3,2)	*(0,1,1,0)*	$\frac{x_3}{x_1}\frac{x_4}{x_2}$	*1*		
		$F(3,1)$	*(0,0,1,0)*	$\frac{x_3}{x_2}\frac{x_4}{x_3}$	*1*		
	$(5,3)$	*0*	*(0,0,0,0)*	1	*1*	$\frac{x_1}{x_1x_2}$	$\frac{x_1}{x_1x_2}\left(1+\frac{x_1x_2^2x_3}{x_1^2}+\frac{x_2x_3}{x_1x_2}\right)$ $=x_1x_2x_3\left(\frac{1}{x_1^2}+\frac{1}{x_1x_2}\left(\sum_{k=1}^{2}\frac{1}{x_kx_{k+1}}\right)\right)$
		F(5,3)	*(2,1,0,0)*	$\frac{x_1x_2}{x_1}\frac{x_2x_3}{x_1}$	*1*		
		F(4,1)	*(0,1,0,0)*	$\frac{x_2}{x_1}\frac{x_3}{x_2}$	*1*		

4 Mutation relations

We recall a notation from [BMV]. For any indecomposable object $X = (a, b) \in \Gamma_n$, the wing determined by X will mean the set of indecomposable objects whose position in the AR-quiver is in the triangle which has X on top, that is, (a', b') such that $a' \geq a$ and $a' + b' \leq a + b$. By Proposition 2.6 in [BMV], every maximal rigid object in Γ_n is in the wing of some $(a, n-1)$ and there is a natural bijection between the set of maximal rigid objects of Γ_n inside the wing of $(a, n-1)$ and the set of tilting modules over path algebra $k\vec{A}_{n-1}$ of the linear quiver of type A_{n-1}. Hence using the complete description of basic tilting modules of the linear quiver of type A in [HR], one can easily check the following proposition for basic maximal rigid objects in Γ_n. Here and what follows, $(a, 0)$ will mean the zero object in Γ_n if $(a, 0)$ appears.

Proposition 4.1. *Let R be a basic maximal rigid object in Γ_n and (a, b) be one of the indecomposable summands of R. Then there is h with $1 \leq h \leq b$, such that $(a, h-1)$ and $(a+h, b-h)$ belong to addR, and every indecomposable summand of R in the wing of (a, b) which is not isomorphic to (a, b) belongs to the wing of $(a, h-1)$ or to the wing of $(a+h, b-h)$. When $b \neq n-1$, there exists an i, $1 \leq i \leq n-b-1$, such that either $(a, b+i)$ or $(a-i, b+i)$ is in addR.*

Proof. The proof follows directly from the proof of the lemma in Section 4.1 in [HR] and the discussion after the proof there. □

With this proposition and Theorem 1.1 in [BMV], we can give all exchange triangles in Γ_n.

Lemma 4.2. *Given two basic maximal rigid objects $T' \oplus \overline{R}$ and $T'' \oplus \overline{R}$ in Γ_n such that both T' and T'' are indecomposable. Then $dimExt^1(T', T'') = 1$ or 2.*
If $dimExt^1(T', T'') = 2$, then T', T'' are of the form $(a, n-1)$. Denote T', T'' by $(a, n-1), (a+h, n-1)$ respectively where $1 \leq a \leq n$, $1 \leq h \leq n-1$. Then the exchange triangles are of the following forms:

$$(a, n-1) \to (a+h, n-h-1) \oplus (a+h, n-h-1) \to (a+h, n-1) \to (a, n-1)[1],$$

$$(a+h, n-1) \to (a, h-1) \oplus (a, h-1) \to (a, n-1) \to (a+h, n-1)[1]$$

If $dimExt^1(T', T) = 1$, then T', T'' are of the form (a, b) with $1 \leq b < n-1$. Denote T' by (a, b). Then T'' is $(a+h, b-h+i)$, where $1 \leq a \leq n$, $1 \leq b \leq n-2$, $1 \leq h \leq b$, $1 \leq i \leq n-b-1$; and *the exchange triangles are of the following forms:*

$$(a,b) \to (a,b+i) \oplus (a+h,b-h) \to (a+h,b-h+i) \to (a,b)[1],$$

$$(a+h,b-h+i) \to (a+b+1,i-1) \oplus (a,h-1) \to (a,b) \to (a+h,b-h+i)[1]$$

Proof. Let $(a,n-1) \oplus \overline{R}$ be a basic maximal rigid object, $1 \le a \le n$. Denote $(a,n-1)$ by T'. It follows from Proposition 4.1 that there is an h, $1 \le h \le n-1$, such that $(a,h-1)$ and $(a+h,n-h-1)$ are in $\mathrm{add}((a,n-1) \oplus \overline{R})$ and other indecomposable summands of $\overline{R}$ are in the wing of $(a,h-1)$ or in the wing of $(a+h,n-h-1)$. It is easy to see that $\mathrm{dimExt}^1(T',X) = 0$ when X belongs to the wings of $(a,h-1)$ or of $(a+h,n-h-1)$. Then $((a+h,n-1)) \oplus \overline{R}$ is a basic maximal rigid object. There is no nonzero map from the wing of $(a+h,n-h-1)$ to $(a,n-1)$ and any maps from the wing of $(a,h-1)$ to $(a,n-1)$ factor from $(a,h-1)$. Since $\mathrm{dimHom}((a,h-1),(a,n-1)) = 2$, $\mathrm{Hom}((a,h-1),(a,n-1))$ contains a non-zero $\mathcal{D}-$map g and a non-zero $\mathcal{T}-$map f. Then $\mathrm{Hom}((a,h-1),(a,n-1))$ is not a cyclic $\mathrm{End}_\Gamma((a,h-1))-$module, otherwise we can assume that $h \in \mathrm{Hom}((a,h-1),(a,n-1))$ is the generator with $h = \alpha_1 f + \alpha_2 g$, $\alpha_i \in k$. Then $g = h\lambda_1 = \alpha_1 f\lambda_1 + \alpha_2 g\lambda_1$, $f = h\lambda_2 = \alpha_1 f\lambda_2 + \alpha_2 g\lambda_2$. Since the composite of a $\mathcal{T}-$map with a $\mathcal{T}-$map is again a $\mathcal{T}-$map, the composite of a $\mathcal{T}-$map with a $\mathcal{D}-$map is a $\mathcal{D}-$map, and any non-zero endomorphism of $(a,h-1)$ is an isomorphism, we have that $\alpha_i = 0$ for $i = 1,2$. This is a contradiction. Therefore the minimal right $\mathrm{add}\overline{R}-$approximation of $(a,n-1)$ is $(a,h-1) \oplus (a,h-1) \overset{(f,g)}{\to} (a,n-1)$. Similarly, we can get the minimal left $\mathrm{add}\overline{R}-$approximation of $(a,n-1)$:$(a,n-1) \to (a+h,n-h-1) \oplus (a+h,n-h-1)$. By Theorem 1.1 in [BMV], we have exchange pair (T',T''). The exchange triangles must be the followings:

$$(a,n-1) \to (a+h,n-h-1) \oplus (a+h,n-h-1) \to (a+h,n-1) \to (a,n-1)[1],$$

$$(a+h,n-1) \to (a,h-1) \oplus (a,h-1) \to (a,n-1) \to (a+h,n-1)[1].$$

In this case, $\mathrm{dimExt}^1_{\Gamma_n}((a,n-1),(a+h,n-1)) = 2$, $1 \le h \le n-1$.
Let $(a,b) \oplus \overline{R}$ be a basic maximal rigid object with $1 \le a \le n$, $1 \le b \le n-2$. Then by Proposition 4.1, there is an i, $1 \le i \le n-b-1$, such that either $(a,b+i)$ or $(a-i,b+i)$ is in $\mathrm{add}(\overline{R})$. Assume that $(a,b+i)$ is in $\mathrm{add}(\overline{R})$ and the number i is the minimal one with respect to property that $(a,b+i) \in \mathrm{add}(\overline{R})$. Then in the wing of (a,b), there is h with $1 \le h \le b$, such that $(a+h,b-h)$ and $(a,h-1)$ in $\mathrm{add}(\overline{R})$. By Proposition 2.6 in [BMV], we have that $((a+h,b-h+i)) \oplus \overline{R}$ is a basic maximal rigid object. Since the object $(a+h,b-h+i)$ satisfies that $a+b \le a+h+b-h+i \le a+n-1$ and $a \le a+h-1 \le a+b-1$, we have that $\mathrm{dimExt}((a,b),(a+h,b-h+i)) = \mathrm{dimHom}((a,b),(a+h-1,b-h+i)) = 1$. There is a non-zero $\mathcal{T}-$map $f : (a,b) \to (a+h,b-h+i)[1]$, which is a part of non-split triangle $(a+h,b-h+i) \to (a+b+1,i-1) \oplus (a,h-1) \to (a,b) \to (a+b1,i-1)[1]$. It is one of exchange triangles between (a,b) and $(a+h,b-h+i)$. Since $(a,b) \to (a,b+i) \oplus (a+h,b-h) \to (a+h,b-h+i) \to (a,b)[1]$ is a non-split triangles in $D^b(\mathcal{T}_n)$, its image under the canonical projective functor is another exchange triangle between (a,b) and $(a+h,b-h+i)$.
Assume that $(a-i,b+i)$ is in $\mathrm{add}(\overline{R})$, and the number i is the minimal one with respect to property that $(a-i,b+i) \in \mathrm{add}(\overline{R})$. Then in the wing of (a,b), there is h with $1 \le h \le b$, such that $(a+h,b-h)$ and $(a,h-1)$ in $\mathrm{add}(\overline{R})$. Similar as above, we have that $(a-i,h+i-1) \oplus \overline{R}$ is a maximal rigid object. Rewrite $(a-i,h+i-1) = (a',b')$, $i' = b-h+1$, then $(a'+i',b') = (a-i,b+i) \in \mathrm{add}(\overline{R})$. We go back to the previous case. Then the exchange triangles we give are all exchange triangles. □

The following theorem is the main result in this section.

Theorem 4.3. *Given two basic maximal rigid objects $R' \oplus \overline{R}$ and $R'' \oplus \overline{R}$ in Γ_n such that both R' and R'' are indecomposable. Then $X_{R'}X_{R''} = X_E + X_{E'}$, where $R' \to E \to R'' \to R'[1]$ and $R'' \to E' \to R' \to R''[1]$ are the exchange triangles.*

Proof. We divide the proof into two cases according to the dimension of extension space between R' and R''.

1. The case of $\dim\mathrm{Ext}^1(R', R'') = 2$. By Lemma 4.2, we assume that $R' = (a, n-1)$, $R'' = (a+h, n-1)$, $E = (a+h, n-h-1) \oplus (a+h, n-h-1)$ and $E' = (a, h-1) \oplus (a, h-1)$, where $1 \le a \le n$, $1 \le h \le n-1$. We only need to consider two subcases:

   (a) $a = 1$. In this case, $(1, n-1), (1, h-1) \in O$, $(1+h, n-h-1) \in I$ and $(1+h, n-1) \in II$. Then by Theorem 3.7,

$$\begin{aligned}
X_{(1,n-1)} &= x_1,\\
X_{(1+h,n-1)} &= x_1 x_{n-h+1} x_{n-h+1}\left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-h}\frac{1}{x_k x_{k+1}}\right)\left(\sum_{l=1}^{n-h}\frac{1}{x_l x_{l+1}}\right)\right),\\
X_{(1+h,n-h-1)} &= x_{n-h+1} x_1\left(\sum_{k=1}^{n-h}\frac{1}{x_k x_{k+1}}\right),\\
X_{(1,h-1)} &= x_{n-h+1}.
\end{aligned}$$

   Hence $X_{(1,n-1)}X_{(1+h,n-1)} = X^2_{(1,h-1)} + X^2_{(1+h,n-h-1)}$.

   (b) $2 \le a \le n-1$. One can assume $1 \le h \le n-a$, otherwise we can replace $(a, n-1)$ by another complement of $\overline{R}$. In this case, $(a, h-1) \in I$ and $(a, n-1), (a+h, n-1), (a+h, n-h-1) \in II$. Then by Theorem 3.7,

$$\begin{aligned}
X_{(a,n-1)} &= x_1 x_{n-a+2} x_{n-a+2}\left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1}\frac{1}{x_k x_{k+1}}\right)\left(\sum_{l=1}^{n-a+1}\frac{1}{x_l x_{l+1}}\right)\right),\\
X_{(a+h,n-1)} &= x_1 x_{n-a-h+2} x_{n-a-h+2}\left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a-h+1}\frac{1}{x_k x_{k+1}}\right)\left(\sum_{l=1}^{n-a-h+1}\frac{1}{x_l x_{l+1}}\right)\right),\\
X_{(a+h,n-h-1)} &= x_1 x_{n-a-h+2} x_{n-a+2}\left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a-h+1}\frac{1}{x_k x_{k+1}}\right)\left(\sum_{l=1}^{n-a+1}\frac{1}{x_l x_{l+1}}\right)\right),\\
X_{(a,h-1)} &= x_{n-a+2} x_{n-a-h+2}\left(\sum_{k=n-a-h+2}^{n-a+1}\frac{1}{x_k x_{k+1}}\right).
\end{aligned}$$

   Hence $X_{(a,n-1)}X_{(a+h,n-1)} = X^2_{(a,h-1)} + X^2_{(a+h,n-h-1)}$.

2. The case of $\dim\mathrm{Ext}^1(R', R'') = 1$. By Lemma 4.2, we assume that $R' = (a, b)$, $R'' = (a+h, b-h+i)$, $E = (a, b+i) \oplus (a+h, b-h)$ and $E' = (a+b+1, i-1) \oplus (a, h-1)$, where $1 \le a \le n$, $1 \le b \le n-2$, $1 \le h \le b$, $1 \le i \le n-b-1$. We need to determine the images of R', R'', and the indecomposable direct summands of E, E' under the map $X_?$. According to Theorem 3.7, we need to know their positions.

(a) $(a,b) \in O$, i.e. $a = 1$. In this case, $(1,b),(1,b+i),(1,h-1) \in O$, $(1+h,b-h),(1+h,b-h+i),(1+b+1,i-1) \in I$. Then by Theorem 3.7,

$$\begin{aligned}
X_{(1,b)} &= x_{n-b},\\
X_{(1+h,b-h+i)} &= x_{n-h+1}x_{n-b-i}\left(\sum_{k=n-b-i}^{n-h}\frac{1}{x_kx_{k+1}}\right),\\
X_{(1,b+i)} &= x_{n-b-i},\\
X_{(1+h,b-h)} &= x_{n-h+1}x_{n-b}\left(\sum_{k=n-b}^{n-h}\frac{1}{x_kx_{k+1}}\right),\\
X_{(1+b+1,i-1)} &= x_{n-b}x_{n-b-i}\left(\sum_{k=n-b-i}^{n-b-1}\frac{1}{x_kx_{k+1}}\right),\\
X_{(1,h-1)} &= x_{n-h+1}.
\end{aligned}$$

Hence $X_{(1,b)}X_{(1+h,b-h+i)} = X_{(1,b+i)}X_{(1+h,b-h)} + X_{(1+b+1,i-1)}X_{(1,h-1)}$.

(b) $(a,b) \in I$. Then $(a+h,b-h),(a,h-1) \in I$. By Theorem 3.7, we have that

$$\begin{aligned}
X_{(a,b)} &= x_{n-a+2}x_{n-a-b+1}\left(\sum_{k=n-a-b+1}^{n-a+1}\frac{1}{x_kx_{k+1}}\right),\\
X_{(a+h,b-h)} &= x_{n-a-h+2}x_{n-a-b+1}\left(\sum_{k=n-a-b+1}^{n-a-h+1}\frac{1}{x_kx_{k+1}}\right),\\
X_{(a,h-1)} &= x_{n-a+2}x_{n-a-h+2}\left(\sum_{k=n-a-h+2}^{n-a+1}\frac{1}{x_kx_{k+1}}\right).
\end{aligned}$$

There are three cases:

i. $a+b < n$, $1 \le i \le n-a-b$. In this case, $(a+h,b-h+i),(a,b+i),(a+b+1,i-1) \in I$. Then by Theorem 3.7,

$$\begin{aligned}
X_{(a+h,b-h+i)} &= x_{n-a-h+2}x_{n-a-b-i+1}\left(\sum_{k=n-a-b-i+1}^{n-a-h+1}\frac{1}{x_kx_{k+1}}\right),\\
X_{(a,b+i)} &= x_{n-a+2}x_{n-a-b-i+1}\left(\sum_{k=n-a-b-i+1}^{n-a+1}\frac{1}{x_kx_{k+1}}\right),\\
X_{(a+b+1,i-1)} &= x_{n-a-b+1}x_{n-a-b-i+1}\left(\sum_{k=n-a-b-i+1}^{n-a-b}\frac{1}{x_kx_{k+1}}\right).
\end{aligned}$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

ii. $a+b < n$, $n-a-b+1 \le i \le n-b-1$. In this case, $(a,b+i),(a+h,b-h+i),(a+b+1,i-1) \in II$. Then by Theorem 3.7,

$$\begin{aligned}
X_{(a+h,b-h+i)} &= x_1x_{n-a-h+2}x_{2n-a-b-i+1}\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{n-a-h+1}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{l=1}^{2n-a-b-i}\frac{1}{x_lx_{l+1}}\right)\right),\\
X_{(a,b+i)} &= x_1x_{n-a+2}x_{2n-a-b-i+1}\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{n-a+1}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{l=1}^{2n-a-b-i}\frac{1}{x_lx_{l+1}}\right)\right),\\
X_{(a+b+1,i-1)} &= x_1x_{n-a-b+1}x_{2n-a-b-i+1}\left(\frac{1}{x_1^2}+\left(\sum_{k=1}^{n-a-b}\frac{1}{x_kx_{k+1}}\right)\left(\sum_{l=1}^{2n-a-b-i}\frac{1}{x_lx_{l+1}}\right)\right).
\end{aligned}$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

iii. $a + b = n$, $1 \le i \le n - b - 1$. In this case, $(a + b + 1, i - 1) = (1, i - 1) \in O$, $(a, b + i), (a + h, b - h + i) \in II$. Then by Theorem 3.7,

$$X_{(a+h,b-h+i)} = x_1 x_{n-a-h+2} x_{2n-a-b-i+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a-h+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b-i} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a,b+i)} = x_1 x_{n-a+2} x_{2n-a-b-i+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b-i} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a+b+1,i-1)} = X_{(1,i-1)} = x_{n-i+1}.$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

(c) $(a, b) \in II$. Then $(a + b + 1, i - 1) = (a + b - n + 1, i - 1) \in I$, $(a, b + i) \in II$. By Theorem 3,7, we have that

$$X_{(a,b)} = x_1 x_{n-a+2} x_{2n-a-b+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a,b+i)} = x_1 x_{n-a+2} x_{2n-a-b-i+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b-i} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a+b+1,i-1)} = X_{(a+b-n+1,i-1)} = x_{2n-a-b+1} x_{2n-a-b-i+1} \left(\sum_{k=2n-a-b-i+1}^{2n-a-b} \frac{1}{x_k x_{k+1}} \right).$$

There are three cases:

i. $1 \le h \le n - a$. In this case, $(a, h - 1) \in I$ and $(a + h, b - h), (a + h, b - h + 1) \in II$. Then

$$X_{(a+h,b-h+i)} = x_1 x_{n-a-h+2} x_{2n-a-b-i+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a-h+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b-i} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a+h,b-h)} = x_1 x_{n-a-h+2} x_{2n-a-b+1} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a-h+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-b} \frac{1}{x_l x_{l+1}} \right) \right),$$

$$X_{(a,h-1)} = x_{n-a+2} x_{n-a-h+2} \left(\sum_{k=n-a-h+2}^{n-a+1} \frac{1}{x_k x_{k+1}} \right).$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

ii. $n - a + 1 < h \le b$. In this case, $(a + h, b - h) = (a + h - n, b - h), (a + h, b - h + i) = (a + h - n, b - h + i) \in I$, $(a, h - 1) \in II$. Then by Theorem 3.7,

$$X_{(a+h,b-h+i)} = X_{(a+h-n,b-h+i)} = x_{2n-a-h+2} x_{2n-a-b-i+1} \left(\sum_{k=2n-a-b-i+1}^{2n-a-h+1} \frac{1}{x_k x_{k+1}} \right),$$

$$X_{(a+h,b-h)} = X_{(a+h-n,b-h)} = x_{2n-a-h+2} x_{2n-a-b+1} \left(\sum_{k=2n-a-b+1}^{2n-a-h+1} \frac{1}{x_k x_{k+1}} \right),$$

$$X_{(a,h-1)} = x_1 x_{n-a+2} x_{2n-a-h+2} \left(\frac{1}{x_1^2} + \left(\sum_{k=1}^{n-a+1} \frac{1}{x_k x_{k+1}} \right) \left(\sum_{l=1}^{2n-a-h+1} \frac{1}{x_l x_{l+1}} \right) \right).$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

iii. $h = n - a + 1$. In this case, $(a + h, b - h) = (1, a + b - n - 1), (a + h, b - h + i) = (1, a + b + i - n - 1) \in O$, $(a, h - 1) = (a, n - a) \in I$. Then

$$X_{(a+h,b-h+i)} = X_{(1,a+b+i-n-1)} = x_{2n-a-b-i+1},$$
$$X_{(a+h,b-h)} = X_{(1,a+b-n-1)} = x_{2n-a-b+1},$$
$$X_{(a,h-1)} = X_{(a,n-a)} = x_{n-a+2}x_1 \left(\sum_{k=1}^{n-a+1} \frac{1}{x_k x_{k+1}} \right).$$

Hence $X_{(a,b)}X_{(a+h,b-h+i)} = X_{(a,b+i)}X_{(a+h,b-h)} + X_{(a+b+1,i-1)}X_{(a,h-1)}$.

Therefore $X_{R'}X_{R''} = X_E + X_{E'}$. □

For the fixed maximal rigid object $T = \oplus_{i=1}^{n-1} T_i$, where $T_i = (1, n - i)$, from Lemma 4.2, we have that the matrix A_T associated to T is $A_T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -2 & 0 & 1 & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & & \cdots & -1 & 0 \end{pmatrix}$. Its Cartan counterpart is of type C_{n-1} (compare to Proposition 3.4 in [BMV]).

Let $\mathrm{indr}\Gamma_n$ be the set of isoclasses of indecomposable rigid objects in Γ_n, and χ_{A_T} be the set of cluster variables of skew-symmetrizable matrix A_T associated to T. Then we have the following:

Theorem 4.4. *The map $X_? : \mathrm{indr}\Gamma_n \longrightarrow \chi_{A_T} : M \mapsto X_M$ is a bijection. This bijection induces a bijection between the set of isoclasses of basic maximal rigid objects in Γ_n and the clusters of the cluster algebra of type C_{n-1}. Furthermore the algebra generated by all X_M, where M runs through $\mathrm{indr}\Gamma_n$, is isomorphic to the cluster algebra of type C_{n-1}.*

Proof. Recall that $T = \oplus_{i=1}^{i=n-1} T_i$. From the definition of X_M and Theorem 3.7, we have that $X_{T_i} = x_i$ for all i. Then $X_?$ sends the couple $(\{T_1, \cdots, T_{n-1}\}, A_T)$ to the initial seed $(\{x_1, \cdots, x_{n-1}\}, A_T)$ of the cluster algebra $\mathcal{A}_{A_T}$. Any basic maximal rigid object can be obtained from the maximal rigid object $\tau^i T$ for some i by iterated mutations, and any $\tau^i T$ can be obtained from T by iterated mutations. Therefore the exchange graph of maximal rigid objects in Γ_n is connected. Then by Theorem 4.3, $X_?(\mathrm{indr}\Gamma_n) \subset \chi_{A_T}$ and $X_? : \mathrm{indr}\Gamma_n \longrightarrow \chi_{A_T}$ is surjective. It follows from Theorem 3.7 that the denominators of X_M are different for all indecomposable rigid objects. Then $X_? : \mathrm{indr}\Gamma_n \longrightarrow \chi_{A_T}$ is injective. Thus the first statement holds. The second statement also follows from the connection of the exchange graph of maximal rigid objects in Γ_n and that the initial maximal rigid object T goes to inial seed $(\{x_1, \cdots, x_{n-1}\}, A_T)$ of the cluster algebra $\mathcal{A}_{A_T}$. The final statement is direct consequence. □

5 Cluster complex of type C

In this section, we use the results proved in Sections 3, 4 to prove the combinatorics of indecomposable rigid objects in Γ_n encodes the cluster combinatorics of the root system of type C and type B. Cluster complexes were defined in [FZ2] for finite root systems. They were realized by quiver representations via decorated representations [MRZ], and later via cluster categories of the corresponding quivers [BMRRT][Z]. Combining with the geometric description of the cluster complex of the root system of type B [FZ2], Buan-Marsh-Vatne [BMV] give a realization of this cluster complex via cluster tubes.

We recall the cluster complex associated to any finite root system from [FZ2]. Let Φ be any finite root system with simple roots $\alpha_1, \cdots, \alpha_n$ and $\Phi_{\geq -1}$ the set of almost positive roots in Φ, i.e. the union of positive roots with negative simple roots.

Fomin and Zelevinsky [FZ2] define a function $(- \parallel -) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$, called the compatibility degree. A pair of roots α, β in $\Phi_{\geq -1}$ are compatible if $(\alpha \parallel \beta) = 0$. The cluster complex $\Delta(\Phi)$ associated to Φ is a simplicial complex, the set of vertices is $\Phi_{\geq -1}$ and the simplices are mutually compatible subsets of $\Phi_{\geq -1}$. This combinatorial object has many interesting properties and applications, we refer to the survey [FR] for further reading.

For the cluster tube Γ_n, we call a set of indecomposable objects a rigid subset provided the direct sum of all indecomposable objects in this set is rigid. Now we define a simplicial complex associated to the cluster tube Γ_n. We always assume $n > 1$ throughout this section.

Definition 5.1. *Let Γ_n be the cluster tube of rank n. The cluster complex $\Delta(\Gamma_n)$ associated to Γ_n is a simplicial complex whose vertices are the isoclasses of indecomposable rigid objects and whose simplices are the the isoclasses of rigid subsets of Γ_n.*

Now fix a root system Φ^C of type C_{n-1}, In [FZ1], Fomin-Zelevinsky gave a bijection from the set of cluster variables of the cluster algebras of type C_{n-1} to $\Phi^C_{\geq -1}$ (they gave this bijection for all finite root systems). Under this bijection, when a cluster variable y is expressed as

$$y = \frac{P(x)}{x^{\alpha}}$$

where P is a polynomial which is not divisible by x_i for every i, then the corresponding almost positive root is α. Therefore, combining this bijection with the bijective map $X_?$ to from $\text{indr}\Gamma_n$ to χ_{A_T} in Section 3, we have a bijection from $\Phi^C_{\geq -1}$ to $\text{indr}\Gamma_n$. This map is denoted by M_T. So, for any $\alpha \in \Phi^C_{\geq -1}$, we denote the object in Γ_n corresponding to α by this bijection by $M_T(\alpha)$.

The main theorem in this section is the following:

Theorem 5.2. *Let Φ^C be the root system of type Γ_{n-1}. Then the map M_T induces an isomorphism from the cluster complex $\Delta(\Phi^C)$ to the cluster complex $\Delta(\Gamma_n)$, which sends vertices to vertices, and simplices to simplices.*

To prove the theorem, we need some preparation:

Definition 5.3. *For any two almost positive roots $\alpha, \beta \in \Phi^C_{\geq -1}$, we define the $T-$compatibility degree $(\alpha \parallel \beta)_T$ of α, β by*

$$(\alpha \parallel \beta)_T = \frac{dimExt^1(M_T(\alpha), M_T(\beta))}{dimEnd(M_T(\alpha))}.$$

As in [FZ2], let σ_i be the permutation of $\Phi^C_{\geq -1}$ defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & if \alpha = -\alpha_j, j \neq i \\ s_i(\alpha) & otherwise \end{cases}$$

where s_i is the the Coxeter generator of the Weyl group of Φ^C corresponding to i. We denote the Coxeter element $\sigma_1 \cdots \sigma_{n-1}$ in the Coxeter group of the root system Φ^C of type C_{n-1} by R (compare [Z]).

Lemma 5.4. *For any $\alpha \in \Phi^C_{\geq -1}$, $M_T(R\alpha) = \tau M_T(\alpha)$.*

Proof. $\mathrm{indr}\Gamma_n = O \bigcup I \bigcup II$ (please see the definition of O, I, II and the Figure.1 in section 3). We divide the proof into several cases according to the position of the object $M_T(\alpha)$.
1. When $M_T(\alpha) = (a, b) \in O$, then $\alpha = -\alpha_{n-b}$. We have $R\alpha = \sigma_1 \cdots \sigma_{n-1}(\alpha) = \sum_{i=1}^{n-b} \alpha_i$. Hence $M_T(R\alpha) = (n, n-b) = \tau(1, n-b) = \tau M_T(\alpha)$.
2. When $M_T(\alpha) = (a, b) \in I$, then $\alpha = \sum_{i=n-a-b+2}^{n-a+1} \alpha_i$. If $a \neq 2$, then $R\alpha = \sum_{i=n-a-b+3}^{n-a+2} \alpha_i$. Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$. If $a = 2$, then $R\alpha = -\alpha_{n-b}$. Hence $M_T(R\alpha) = (1, b) = \tau(2, b) = \tau M_T(\alpha)$.
3. When $M_T(\alpha) = (2, n-1) \in II$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-1} \alpha_i$. We have $R\alpha = -\alpha_1$. Hence $M_T(R\alpha) = (1, n-1) = \tau(2, n-1) = \tau M_T(\alpha)$.
4. When $M_T(\alpha) = (a, n-1) \in II$ with $3 \leq a \leq n$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+1} \alpha_i$. We have $R\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+2} \alpha_i$. Hence $M_T(R\alpha) = (a-1, n-1) = \tau(a, n-1) = \tau M_T(\alpha)$.
5. When $M_T(\alpha) = (a, b) \in II$ with $a + b = n + 1$ and $2 \leq b \leq n-2$, then $\alpha = \alpha_1 + 2\sum_{i=1}^{n-a+1} \alpha_i + \sum_{i=n-a+2}^{2n-a-b} \alpha_i$. It follows $R\alpha = \sum_{i=2}^{n-a+2} \alpha_i$. Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.
6. When $M_T(\alpha) = (n, 1) \in II$, then $\alpha = \sum_{i=1}^{n-1} \alpha_i$. It follows $R\alpha = \alpha_2$. Hence $M_T(R\alpha) = (n-1, 1) = \tau(n, 1) = \tau M_T(\alpha)$.
7. When $M_T(\alpha) = (a, b) \in II$ with $a = n$ and $2 \leq b \leq n-2$, then $\alpha = \sum_{i=1}^{n-b} \alpha_i$. It follows $R\alpha = \alpha_1 + 2\alpha_2 + \sum_{i=3}^{n-b+1} \alpha_i$. Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.
8. When $M_T(\alpha) = (a, b) \in II$ with $a < n, 1 < b < n-1$ and $a + b > n + 1$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+1} \alpha_i + \sum_{i=n-a+2}^{2n-a-b} \alpha_i$. It follows $R\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+2} \alpha_i + \sum_{i=n-a+3}^{2n-a-b+1} \alpha_i$. Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.
The proof of this lemma is completed.

□

Using the dimension formulas in the proof of Theorem 3.7, we have the following fact:

Lemma 5.5. *For any positive root β and any i, $[\beta : \alpha_i] = \frac{dimHom(T_i[-1], M_T(\beta))}{dimEnd(T_i)}$.*

Lemma 5.6. *The $T-$compatibility degree satisfies the following conditions:*

$$(-\alpha_i \parallel \beta)_T = \max([\beta : \alpha_i], 0), \tag{1}$$

$$(R\alpha \parallel R\beta)_T = (\alpha \parallel \beta)_T, \tag{2}$$

for any $\alpha, \beta \in \Phi^C_{\geq -1}$, any $1 \leq i \leq n-1$.

Proof. By the definition, $(-\alpha_i \parallel \beta)_T = \frac{\mathrm{dimExt}^1(T_i, M_T(\beta))}{\mathrm{dimEnd}(T_i)} = \frac{\mathrm{dimHom}(T_i[-1], M_T(\beta))}{\mathrm{dimEnd}(T_i)}$. Then by Lemma 5.5 it equals $[\beta : \alpha_i]$ if β is a positive root, or 0 otherwise. This proves that (1) holds. From Lemma 5.4, we have $(R\alpha \parallel R\beta)_T = \frac{\mathrm{dimExt}^1_\Gamma(M_T(R\alpha), M_T(R\beta))}{\mathrm{dimEnd}_\Gamma(M_T(R\alpha))} = \frac{\mathrm{dimExt}^1_\Gamma(\tau M_T(\alpha), \tau M_T(\beta))}{\mathrm{dimEnd}_\Gamma(\tau M_T(\alpha))} = \frac{\mathrm{dimExt}^1_\Gamma(M_T(\alpha), M_T(\beta))}{\mathrm{dimEnd}_\Gamma(M_T(\alpha))} = (\alpha \parallel \beta)_T$. This proves that (2) holds.

□

Proof. of Theorem : From Lemmas 5.5, 5.6, the compatibility degree $(-\|-)_T$ is the same as $(-\|-)$ in [FZ2]. It follows that α, β are compatible if and only if $M_T(\alpha), M_T(\beta)$ form a rigid subset. Therefore M_T induces the desired bijection from $\Delta(\Phi^C)$ to $\Delta(\Gamma_n)$.

□

Let Φ^B be the root system of type B_{n-1}. Then $\Phi^B_{\geq -1}$ is the dual of $\Phi^C_{\geq -1}$ via $\alpha \mapsto \alpha^\vee$. Then $(\alpha \parallel \beta) = (\beta^\vee \parallel \alpha^\vee)$ (Proposition 3.15 [FZ2]). Then we have the following corollary (compare Theorem 3.5. in [BMV]).

Corollary 5.7. *Let Φ^B be the root system of type B_{n-1}. Then the cluster complex $\Delta(\Phi^B)$ of Φ of type B_{n-1} is isomorphic to $\Delta(\Gamma_n)$.*

ACKNOWLEDGMENTS.

The authors would like to thank Aslak Bakke Buan, Bin Li, Yann Palu and Dong Yang for their interesting, helpful discussion and suggestions.

References

[Am] C.Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. Ann. Inst. Fourier **59**, 2525-2590, 2009.

[BKL1] M.Barot, D.Kussin and H.Lenzing. The Grothendieck group of a cluster category. Jour. Pure and App. Algebra. **212**, 33-46, 2008.

[BKL2] M.Barot, D.Kussin and H.Lenzing. The cluster category of a canonical algebra. Trans. Amer. Math. Soc. **362**, no. 8, 4313–4330, 2010.

[BIKR] I.Burban, O.Iyama, B.Keller and I.Reiten. Cluster tilting for one-dimensional hypersurface singularities. Adv. Math. **217**, 2443-2484, 2008.

[BIRS] A.Buan, O.Iyama, I.Reiten and J.Scott. Cluster structure for 2−Calabi-Yau categories and unipotent groups. Compo. Math. **145**(4), 1035-1079, 2009.

[BMV] A.Buan, R.Marsh and D.Vatne. Cluster structure from 2−Calabi-Yau categories with loops. Mathematische Zeitschrift **265**(4), 951-970, 2010.

[BMRRT] A.Buan, R.Marsh, M.Reineke, I.Reiten and G.Todorov. Tilting theory and cluster combinatorics. Advances in Math. **204**, 572-618, 2006.

[CC] P.Caldero and F.Chapoton. Cluster algebras as Hall algebras of quiver representations. Comment. Math. Helv. **81**, 595-616, 2006.

[CCS] P.Caldero, F.Chapoton and R.Schiffler. Quivers with relations arising from clusters (A_n case). Transactions of the AMS. **358**, 1347-1364, 2006.

[CK] P.Caldero and B.Keller. From triangulated categories to cluster algebras II. Ann.Sci.Ecole.Norm.Sup.(4) **39**, 983-1009,2006.

[DK] R.Dehy and B.Keller. On the Combinatorics of Rigid Objects in 2-Calabi-Yau Categories. International Mathematics Research Notices, **2008**(11), 17 pages.

[DX] M.Ding and F.Xu. The cluster character for cyclic quivers. arXiv:1001.4360.

[Du1] G.Dupont. An approach to non-simply laced cluster algebras. J. Algebra **320**, no. 4, 1626–1661, 2008.

[Du2] G.Dupont. Cluster multiplication in regular components via generalized Chebyshev polynomials. arXiv:0801.3964v2. Algebra and Representation Theory (in press).

[F] S.Fomin. Total positivity and cluster algebras. arXiv:1005.1086, to appear in Proceedings of ICM 2010.

[FR] S.Fomin and N.Reading. Root systems and generalized associahedra, Geometric Combinatorics (Park City, UT, 2003), 63C131, IAS/Park City Math. Ser., 14, Amer. Math. Soc., Providence, RI, 2007.

[FZ1] S.Fomin and A.Zelevinsky. Cluster Algebras I: Foundations. Jour. Amer. Math. Soc. **15**, no. 2, 497–529, 2002.

[FZ2] S.Fomin and A.Zelevinsky. Y-system and generalized associahedra, Ann of Math. **158**, 977-1018, 2003.

[FZ3] S.Fomin and A.Zelevinsky. Cluster algebras II: Finite type classification. Invent. Math. **154**, no.1, 63-121, 2003.

[FZ4] S.Fomin and A.Zelevinsky. Cluster algebras: Notes for the CDM-03 Conference. CDM 2003: Current Developments in Mathematics. International Press, 2004.

[FuKe] C.Fu and B.Keller. On cluster algebras with coefficients and $2-$Calabi-Yau categories. Trans. Amer. Math. Soc. **362**,859-895, 2010.

[GLS1] C.Geiß, B.Leclerc and J.Shröer. Rigid modules over preprojective algebras. Invent. math. **165**(3), 589-632, 2006.

[GLS2] C.Geiß, B.Leclerc and J.Shröer. Preprojective algebras and cluster algebras. Trends in representation theory of algebras and related topics, 253-283, EMS Ser.Congr.Rep. Eur.Math.Soc. Zürich,2008.

[H] D.Happel. Triangulated categories in the representation theory of quivers. LMS Lecture Note Series, 119. Cambridge, 1988.

[HR] D.Happel, C.M.Ringel. Construction of tilted algebras. Representations of Algebras, 125-144, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

[Hu] J. E. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer-Verlag, New York, Heilelberg, Berlin, 1972.

[I] O.Iyama. Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories. Adv. Math. 210(2007), 22-50.

[IY] O.Iyama and Y.Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. **172**, 117C168, 2008.

[Ke1] B.Keller. Triangulated orbit categories. Documenta Math. **10**, 551-581, 2005.

[Ke2] B.Keller. Cluster algebras, quiver representations and triangulated categories. LMS Lecture Notes Ser., 375, Cambridge, 76–160, 2010.

[Ke3] B.Keller. Triangulated Calabi-Yau categories, Trends in Representation Theory of Algebras (Zurich) (A. Skowronski, ed.), European Mathematical Society, 2008, pp. 467-489.

[KR] B.Keller and I.Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.

[KZ] S.Koenig and B.Zhu. From triangulated categories to abelian categories– cluster tilting in a general framework. Math. Zeit. 258, 143-160, 2008.

[MSW] G.Musiker, R.Schiffler, L.Williams. Positivity for cluster algebras from surfaces arXiv:0906.0748.

[Pa1] Y.Palu. Cluster characters for 2−Calabi-Yau triangulated categories. Ann. Inst. Fourier, **58**, 133-171, 2008.

[Pa2] Y.Palu. Grothendieck group and generalized mutation rule for 2−Calabi-Yau triangulated categories. Jour.of Pure and Appl. Alg. **213**, 1438-1449, 2009.

[Pla1] P.Plamondon. Cluster characters for cluster categories with infinite-dimensional morphism spaces. To appear in Adv. Math., doi: 10.1016/j.aim.2010.12.010. See also arXiv. 1002.4956.

[Pla2] P.Plamondon. Cluster algebras via cluster categories with infinite-dimensional morphism spaces. arXiv. 1004.0830.

[Re] I.Reiten. Cluster categories. arXiv:1012.4949, to appear in Proceedings of ICM 2010.

[Rin] C.M.Ringel. Some remarks concerning tilting modules and tilted algebras. Origin. Relevance. Future. An appendix to the Handbook of tilting theory, edited by Lidia Angeleri-Hügel, Dieter Happel and Henning Krause. LMS Lecture Notes Series 332. Cambridge, 413-472, 2007.

[V] D.Vatne. Endomorphism rings of maximal rigid objects in cluster tubes. arXiv: 0905.1796.

[Y] D.Yang. Endomorphism algebras of maximal rigid objects in cluster tubes. arXiv: 1004.1303.

[ZZ] Y.Zhou and B.Zhu. Maximal rigid subcategories in 2−Calabi-Yau triangulated categories. arXiv:1004.5475.

[Z] B.Zhu BGP-reflection functors and cluster combinatorics. J. Pure Appl. Algebra. **209**, 497-506, 2007.