

Hall type algebras associated to triangulated categories

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- 2 Hall type algebras associated to triangulated categories
- 3 An improvement of Peng-Xiao's theorem

Outline

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Ringel-Hall type algebras I

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where $[X]$ isomorphism class. The multiplication:

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where $[X]$ isomorphism class. The multiplication:

$$[X] * [Y] = \sum_{[L]} g_{XY}^L [L],$$

where the Hall number

$$g_{XY}^L = |\{(f, g) \mid 0 \rightarrow X \xrightarrow{f} L \xrightarrow{g} Y \rightarrow 0\} / \text{Aut } X \times \text{Aut } Y| \in \mathbb{Z}.$$

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New types (potential):

- **Constructible functions on Artin stacks** are used by Kapranov- Vasserot, Joyce.
- **Cluster multiplication formulas**, as by Caldero-Chapoton-Keller, Geiss-Leclerc-Schroerer, Hubery, Xiao-Xu, Palu.

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- The realization of a non-standard positive part of $U_v(\widehat{\mathfrak{sl}}_2)$ via $\mathcal{H}(\text{coh}\mathbb{P}^1)$ by Kapranov (1998).
- Elliptic Lie algebras of type $D_4^{(1,1)}$ and $E_{6,7,8}^{(1,1)}$ by Peng and Lin (2004).

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- Loop algebras of Kac-Moody algebras via Hall algebras of weighted projective lines by Schiffmann (2004).
- A characterization of Drinfeld new presentation of quantum loop algebras via Hall algebras of weighted projective lines by Dou-Jiang-Xiao (2010).
- Counting stability conditions and wall-crossing via holomorphic functions to Hall Lie algebras of stack functions by Joyce (2007).

Ringel-Hall type algebras IV

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illustrated by the diagram

$$\begin{array}{ccccc} X & \longrightarrow & L & \longrightarrow & Y \\ \parallel & & \downarrow & & \downarrow \\ X & \longrightarrow & M & \longrightarrow & L' \\ & & \downarrow & & \downarrow \\ & & Z & \equiv & Z \end{array}$$

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Two interesting classes of triangulated categories

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Let k be a field with $|k| = q$ and \mathcal{C} be a k -additive finitary, Krull-Schmidt triangulated category.

- \mathcal{C} is (left) homologically finite if for $X, Y \in \mathcal{C}$, $\text{Hom}(X[i], Y) = 0$ for $i \gg 0$.
Example: the derived category of an abelian category
- \mathcal{C} is n -periodic if the translation functor $T = [1]$ such that $T^n \cong 1$.
Example: 2-periodic orbit categories of $\mathcal{D}^b(H)$ for hereditary category H as triangulated categories of singularities in homological mirror symmetry.

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Hall type algebras of triangulated categories II

Theorem (Toën, Xiao-Xu)

Let \mathcal{C} be a (left) homologically finite triangulated category and $\mathcal{H}(\mathcal{C})$ be the \mathbb{Q} -space with the basis $\{[X] \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by

$$[X] * [Y] = \sum_{[L]} F_{XY}^L [L],$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $[0]$.

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$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $[0]$. Here

$$F_{XY}^L = \frac{|(X, L)_Y|}{|\mathrm{Aut} X|} \cdot \left(\prod_{i>0} \frac{|\mathrm{Hom}(X[i], L)|^{(-1)^i}}{|\mathrm{Hom}(X[i], X)|^{(-1)^i}} \right) \in \mathbb{Q}.$$

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Theorem (Chen-Xu)

Let \mathcal{C} be an odd periodic triangulated category and $\mathcal{H}(\mathcal{C})$ be the $\mathbb{Q}(v)$ -space with the basis $\{[X] \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by

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$$[X] * [Y] = \sum_{[L]} F_{XY}^L [L],$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $[0]$. Here $q = v^2$ and

$$F_{XY}^L = \frac{|(X, L)_Y|}{|\mathrm{Aut} X|} \cdot \left(\prod_{i=1}^t \frac{|\mathrm{Hom}(X[i], L)|^{(-1)^i}}{|\mathrm{Hom}(X[i], X)|^{(-1)^i}} \right)^{\frac{1}{2}} \in \mathbb{Q}[v, v^{-1}].$$

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- $\mathcal{D}_3 := \mathcal{D}_3(\mathcal{A})$: the 3-periodic derived category;
- Objects in \mathcal{D}_3 :

$$\dot{X} : \dots \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_1 \xrightarrow{f_1} \dots$$

- Indecomposable objects in $\mathcal{D}_3(\mathcal{A})$: $X[i]$ for $X \in \text{ind}\mathcal{A}$ and $i = 0, 1, 2$.

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- There is a Galois covering (dense) functor $F : \mathcal{D} \rightarrow \mathcal{D}_3$.

Examples from Hereditary categories II

$\mathcal{H}(\mathcal{D})$ has the set of generators $\{u_{[X[i]]} \mid i \in \mathbb{Z}, X \in \mathcal{A}\}$ with the following generating relations:

$$\textcircled{1} \quad u_{[X[m]]} \cdot u_{[Y[m]]} = \sum_{[L]} F_{XY}^L \cdot u_{[L[m]]};$$

$$\textcircled{2} \quad u_{[X]^{[m+1]}} \cdot u_{[Y]^{[m]}} = \sum_{[K],[C]} F_{X,Y[1]}^{K[m+1] \oplus C[m]} \cdot u_{[K[m+1]]} \cdot u_{[C[m]]};$$

$$\textcircled{3} \quad u_{[X[n]]} \cdot u_{[Y[m]]} = q^{(-1)^{n-m} \langle Y, X \rangle} u_{[Y[m]]} \cdot u_{[Y[n]]} \text{ if } n > m + 1.$$

Examples from Hereditary categories III

Now $\mathcal{A} = \text{mod}kQ$ for a finite quiver without oriented cycles and a finite field k . $\mathcal{H}(\mathcal{D}_3)$ has the set of generators $\{u_{X[i]} \mid i = 0, 1, 2, X \in \mathcal{A}\}$ with the following generating relations:

① for $n = 0, 1, 2$, $u_{[X[n]]} \cdot u_{[Y[n]]} = \sum_{[L]} F_{XY}^L \cdot u_{[L[n]]}$;

② for $n = 0, 1$,

$$u_{[X[n]]} \cdot u_{[Y[n+1]]} = \sum_{[K],[C]} F_{X,Y[1]}^{K[1] \oplus C} \cdot u_{[K[n+1]]} \cdot u_{[C[n]]}$$
;

③ $u_{[X[2]]} \cdot u_{[Y]} = \sum_{[K],[C]} F_{X,Y[1]}^{K[1] \oplus C} \cdot u_{[K]} \cdot u_{[C[2]]}$.

Hall Lie algebras of triangulated categories I

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Hall Lie algebras of triangulated categories I

still open: How to define Hall algebras for 2-periodic triangulated categories? The partial answer is given by Hall Lie algebras.

Theorem (Peng-Xiao)

Let \mathcal{C} be 2-periodic over a field k with $|k| = q$ and $G(\mathcal{C})$ be the Grothendieck group. For any $M \in \mathcal{C}$, h_M its canonical image in $G(\mathcal{C})$. Set

$$L(\mathcal{C}) = \sum_{[M]; M \in \text{ind}\mathcal{C}} \mathbb{Z}\bar{h}_M / \bigoplus_{[X]; X \in \text{ind}\mathcal{C}} \bigoplus \mathbb{Z}[X]$$

where $d(M) = \dim_k(\text{End}M/\text{rad}\text{End}M)$ and $\bar{h}_M = h_M/d(M)$. Then $L(\mathcal{C})/(q-1)L(\mathcal{C})$ is a Lie algebra over $\mathbb{Z}/(q-1)$.

Note: there is a geometric construction of Lie algebras from 2-periodic derived categories by Xiao-Xu-Zhang.

Hall Lie algebras of triangulated categories II

The multiplication inducing Lie bracket in Peng-Xiao theorem is

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$$F_{XY}^L = |\{(f, g) \mid X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1]\} / \text{Aut} X \times \text{Aut} Y| \in \mathbb{Z}.$$

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Write

$$[X] * [Y]([L]) = F_{XY}^L.$$

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Write

$$[X] * [Y]([L]) = F_{XY}^L.$$

The multiplication is not associative, i.e., it does not hold

$$(((X] * [Y]) * [Z]) - [X] * ([Y] * [Z]))([M]) = 0$$

for any $M \in \mathcal{C}$.

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Question: How to overcome the non-associativity of the multiplication?

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Partial answer:

- Inspired by theorems by Toën, Xiao-Xu and Chen-Xu, F_{XY}^L should be chosen in \mathbb{Q} or $\mathbb{Q}(v)$. Then, use the approach to prove the associativity.

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Partial answer:

- Inspired by theorems by Toën, Xiao-Xu and Chen-Xu, F_{XY}^L should be chosen in \mathbb{Q} or $\mathbb{Q}(v)$. Then, use the approach to prove the associativity.
- Inspired by Lusztig's construction of finite dimensional quotients of enveloping algebras, the multiplication should not be intrinsic.

An improvement of Peng-Xiao's theorem II

For any objects U, V and W in \mathcal{C} , define

- 1 $F_{UV}^W := \frac{|(U, W)_V|}{|\text{Aut}U|}$, if $U \not\cong W \oplus V[1]$,
- 2 $\overline{F}_{UV}^W := \frac{|(W, V)_{U[1]}|}{|\text{Aut}V|}$, if $V \not\cong W \oplus U[1]$,
- 3 $F_{W \oplus V[1], V}^W := |\text{Hom}(W, V[1])| \cdot \frac{|(W \oplus V[1], W)_V|}{|\text{Aut}(W \oplus V[1])|} = \frac{1}{|\text{Aut}V|}$,
- 4 $\overline{F}_{U, W \oplus U[1]}^W := |\text{Hom}(U[1], W)| \cdot \frac{|(W, W \oplus U[1])_{U[1]}|}{|\text{Aut}(W \oplus U[1])|} = \frac{1}{|\text{Aut}U|}$.

An improvement of Peng-Xiao's theorem III

For any $X \in \mathcal{C}$, we introduce a new variant θ_X associated to X . Define a new multiplication between indecomposable objects by the following rules. For $X, Y \in \mathcal{C}$,

- 1 $\theta_X \theta_Y = \theta_Y \theta_X$,
- 2 $\theta_X - \theta_{X[1]} = \bar{h}_X$,
- 3 if $X \not\cong Y[1]$, $[X] * [Y] := \sum_{[L]} F_{YX}^L [L]$,
- 4 $[X] * [X[1]] := \sum_{[L]} F_{X[1]X}^L + \theta_{X[1]}$,
- 5 $[Y] * \theta_X := -\frac{\dim_k \text{Hom}(X[1], Y)}{d(X)}$,
- 6 $\theta_X * [Y] := -\frac{\dim_k \text{Hom}(Y, X)}{d(X)}$.

An improvement of Peng-Xiao's theorem IV

Theorem

With the above notation, in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, we have

$$([Y] \cdot [X]) \cdot [Z] - [Y] \cdot ([X] \cdot [Z])(M) = 0.$$

for X, Y, Z and $M \neq 0$ in $\text{ind}\mathcal{C}$.

We call this property **the almost associativity of the multiplication for indecomposable objects**. The multiplication induces Lie brackets in Peng-Xiao's theorem.

Corollary

The almost associativity \Rightarrow Peng-Xiao's theorem

The idea of the proof of the almost associativity I

Consider the key expression

$$S = \sum_{[L], L \in \mathcal{C}} F_{XY}^L F_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}} F_{ZX}^{L'} F_{L'Y}^M.$$

By definition, it is

$$\sum_{[L]} \frac{|(X, L)_Y|}{|\text{Aut} X|} \cdot \frac{|(M, L)_{Z[1]}|}{|\text{Aut} L|} - \sum_{[L']} \frac{|(L', X)_{Z[1]}|}{|\text{Aut} X|} \cdot \frac{|(L', M)_Y|}{|\text{Aut} L'}.$$

Set

$$\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]} := \left\{ \begin{pmatrix} m \\ f \end{pmatrix} \in \text{Hom}(M \oplus X, L) \mid \right.$$

$$\left. \text{Cone}(f) \simeq Y, \text{Cone}(m) \simeq Z[1] \text{ and } \text{Cone} \begin{pmatrix} m \\ f \end{pmatrix} \simeq L'[1] \right\}$$

The idea of the proof of the almost associativity II

$$\begin{aligned}
 S &= \sum_{[L],[L']} \frac{1}{|\mathrm{Aut} X|} \cdot \delta_{L,M \oplus Z[1]} \cdot \frac{|\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}|}{|\mathrm{Aut} L|} \\
 &\quad - \sum_{[L],[L']} \frac{1}{|\mathrm{Aut} X|} \cdot \delta'_{L',M \oplus Y[1]} \cdot \frac{|\mathrm{Hom}(L', M \oplus X)_L^{Y,Z[1]}|}{|\mathrm{Aut} L'|}
 \end{aligned}$$

where $\delta_{L,M \oplus Z[1]} = |\mathrm{Hom}(Z[1], M)|$ for $L \cong M \oplus Z[1]$ and 1 otherwise, $\delta'_{L',M \oplus Y[1]} = |\mathrm{Hom}(M, Y[1])|$ for $L' \cong M \oplus Y[1]$ and 1 otherwise.

The idea of the proof of the almost associativity III

The equivalence between the octahedral axiom and the pushout \Rightarrow

$$\begin{aligned} \mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]} / \mathrm{Aut} L &= \mathrm{Hom}(L', M \oplus X)_L^{Y, Z[1]} / \mathrm{Aut} L' \\ &:= V(L', L; M \oplus X)_{Y, Z[1]}. \end{aligned}$$

Lemma

$$\frac{|(M, L)_{Z[1]}|}{|\mathrm{Aut} L|} = \sum_{\alpha \in V(Z, L; M)} \frac{|\mathrm{End} L_1(\alpha)|}{|n(\alpha) \mathrm{Hom}(Z[1], L) | |\mathrm{Aut} L_1(\alpha)|}$$

and

$$\frac{|(Z, M)_L|}{|\mathrm{Aut} Z|} = \sum_{\alpha \in V(Z, L; M)} \frac{|\mathrm{End} Z_1(\alpha)|}{|\mathrm{Hom}(Z[1], L) n(\alpha) | |\mathrm{Aut} Z_1(\alpha)|}$$

The idea of the proof of the almost associativity IV

Except two exceptional cases, the sum S is

$$\sum_{[L],[L']} \left(\sum_{\bar{\alpha} \in \overline{V}(L',L;M \oplus X)_{Y,Z[1]}} \left(\frac{1}{q^{s(\alpha)}} - \frac{1}{q^{t(\alpha)}} \right) \cdot \frac{1}{|G_{\alpha}|} \right).$$

Here, G_{α} is the stable subgroup of $\text{Aut} X$ and

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Except two exceptional cases, G_α is a vector space $\Rightarrow S = 0$ over $\mathbb{Z}[\frac{1}{q}]/(q-1)$.