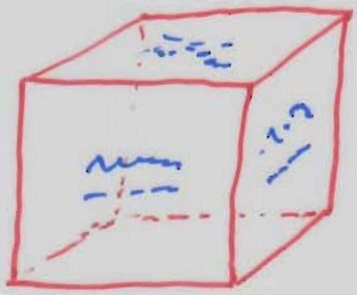
Standard objects and exceptional objects,
and a box full of structure

(Report on a joint project with Sergiy Ovsiyenko)



1. Read the texts on the six sides
(same information –
in different languages)

2. Open the box
(find some structure inside)

I. Standard systems, exceptional sequences and $\mathcal{F}(\Delta)$

algebra	geometry

abelian k-category

k a field

standard system	exceptional sequence
$\Delta = \{\Delta(1), \dots, \Delta(n)\}$	$\Delta = \{\Delta(1), \dots, \Delta(n)\}$
$\forall i$: $\mathrm{End}(\Delta(i)) = k$	$\forall i$: $\mathrm{End}(\Delta(i)) = k$
$\mathrm{Hom}(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$	$\mathrm{Hom}(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$
$\mathrm{Ext}^n(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j \quad \forall n > 0$	$\mathrm{Ext}^n(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j \quad \forall n > 0$

examples: highest weight categories
- Verma modules
- Weyl modules
- Specht modules (char $k \neq 2, 3$)
- ...

quasi-hereditary algebras: A fin dim $/k$, n simples

A is quasi-hereditary $\Leftrightarrow$ $\exists$ standard system with n objects such that ${}_A A \in \mathcal{F}(\Delta) = \{\Delta\text{-filtered objects}\}$

use for instance to translate from geometry to algebra

- $D^b(\mathrm{coh}\,\mathbb{P}^1(\mathbb{C})) \xrightarrow{\sim} D^b(\text{Kronecker algebra})$

  and many other cases (Bondal, Orlov, ...)
- from matrix factorisations to quivers

  [H. Kajiura, K. Saito, A. Takahashi]

II. A_∞-structures and twisted stalks
(related to twisted complexes, Bondal–Kapranov, Kontsevich, Keller, Lefèvre-Hasegawa – – cf Kajiura's talk)
[B. Keller, Introduction to A-infinity algebras and modules, 2001
B. Keller, A-infinity algebras in representation theory, 2002
P. Seidel, Fukaya categories and Picard-Lefschetz theory, 2008]

set $\Delta := \Delta(1) \oplus \dots \oplus \Delta(n)$

$\mathrm{Ext}^*(\Delta,\Delta) := \bigoplus_{m \geq 0} \mathrm{Ext}^m(\Delta,\Delta)$ Yoneda extension algebra

$\overset{\infty}{\mathrm{End}}{}^{\bullet}(\Delta,\Delta)$ is the homology of $\mathrm{Hom}^{\bullet}$(proj res of Δ, proj res of Δ)

A_∞-algebra (or category)

m_2 multiplication is not associative
associator m_3 –
$\rightsquigarrow$ higher multiplications m_n
(m_1 = differential)

Kadeishvili: $\overset{\infty}{\mathrm{End}}{}^{\bullet}(\Delta,\Delta)$ carries an A_∞-structure quasi-isomorphic to that of $\mathrm{Hom}^{\bullet}$(proj res of Δ, proj res of Δ)
"minimal model": $m_1 = 0$, m_2 from Yoneda multiplication

$\mathcal{F}(\Delta)$ can be reconstructed from $\overset{\infty}{\mathrm{End}}{}^{\bullet}(\Delta,\Delta)$
analogue: X module $\Rightarrow$ add $X \simeq$ proj-End(X)
here: $\mathcal{F}(\Delta) \simeq \mathcal{F}(\overset{\infty}{\mathrm{End}}{}^{\bullet}(\Delta,\Delta))$

explicit construction by twisted stalks
similar to twisted complexes

$$\begin{tikzcd} \mathcal{A} \arrow[r, hookrightarrow] \arrow[dr, "\mathrm{Yoneda}"'] & \mathrm{tw}\,\mathcal{A} \arrow[d, "\wr"] \\ & C_\infty \mathcal{A} \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} H^0\,\mathrm{tw}\,\mathcal{A} \arrow[d, "\wr"] \\ \mathrm{Triang} \subset D_\infty \mathcal{A} \end{tikzcd}$$

Quasi-hereditary algebras cover "everything":

Theorem (V. Dlab + C.M. Ringel, 1992):
Given a standard system Δ in an abelian k-category, there exists a qh algebra A such that

$$\mathcal{F}(\Delta) \simeq \mathcal{F}(\Delta(A)) \quad \text{(as exact categories)}$$

Problem: Describe the structure of the exact category $\mathcal{F}(\Delta)$.

example: principal block of BGG-category $\mathcal{O}$ for $sl(2,\mathbb{C})$, up to Morita equivalence

$$A = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{pmatrix} : a,b,c,d,e \in k = \mathbb{C} \right\}$$

$$_AA = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \quad \Delta(1) = 1, \quad \Delta(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$$

$$\mathcal{F}(\Delta): \quad \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \nearrow \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \searrow 1$$

Ringel, 1991: A qh $\Rightarrow$ $\mathcal{F}(\Delta)$ has (relative) almost split sequences

compare $k(\cdot \to \cdot)$-mod:

$$2 \nearrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \searrow 1$$

$$\underset{\text{not abelian}}{\mathcal{F}(\Delta)} \not\simeq \underset{\text{abelian}}{k(\cdot \to \cdot)\text{-mod}}$$

twisted stalks

object in $\mathcal{F}(\Delta)$ is an iterated extension of $\Delta(1), \dots, \Delta(n)$ (ordered)

long exact cohomology sequence $\rightsquigarrow$

data needed: # copies of $\Delta(i)$

elements in $\mathrm{Ext}^1(\Delta(i), \Delta(j))$

$\rightsquigarrow$ matrix δ in add $\mathcal{A}$, strictly upper triangular, degree 1 entries

$\mathcal{A} \overset{\infty}{=} \mathrm{End}^*(\oplus\Delta)$

which matrices occur?

miracle: a matrix δ occurs $\Leftrightarrow$ it satisfies the Maurer – Cartan equation $\sum_{t=1}^{\infty} (-1)^{\frac{t(t-1)}{2}} m_t(\delta, \delta, \dots, \delta) = 0$ (MC)

$\uparrow$ extended to matrices

$\rightsquigarrow$ new A_∞-structure on twisted objects

$$m_1^{tw}(f) = m_1(f) - m_2(\delta', f) + m_2(f, \delta) + m_3(\delta', \delta', f) \pm \dots$$

$f: (B, \delta) \to (B', \delta')$

even degree (odd degree: different signs)

example

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, bend left, "1"] \arrow[r, bend right, "0"'] & \underset{2}{\circ} \end{tikzcd} \qquad \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \end{matrix}$$

twisted stalk

$$\begin{tikzcd} V_1 \circ \arrow[r, bend left, "\text{matrix}"] \arrow[r, bend right] & \circ V_2 \end{tikzcd} \qquad V_1, V_2 \text{ vectorspaces}$$

same objects for $\circ \longrightarrow \circ$ A_2-quiver

but morphisms are different

(matrices with degree 0 entries, satisfying some equation)

III. add $\mathcal{A}$ and conv $\mathcal{A}$

$\mathcal{A}$ A_∞-category, s shift $(s\mathcal{A})^n = \mathcal{A}^{n+1}$

$$\begin{tikzcd}
\mathcal{A}^{\otimes n} \arrow[r, "m_n"] \arrow[d, "s^{\otimes n}"'] \arrow[dr, phantom, "\varepsilon"] & \mathcal{A} \arrow[d, "s"] \\
(s\mathcal{A})^{\otimes n} \arrow[r, "b_n"'] & s\mathcal{A}
\end{tikzcd}$$

b_n higher multiplication
sign-free

add $\mathcal{A}$ (matrices over $\mathcal{A}$) has "same" A_∞-structure

define conv $s\mathcal{A}$: same objects

$$(\mathrm{conv}\, s\mathcal{A})^n(i,j) := \mathrm{Hom}_k\big(\overset{k\text{-dual}}{D}(s\mathcal{A})^{-n}(j,i),\ \mathrm{Hom}_k(X(i), Y(j))\big)$$

on objects X, Y

uses $V \otimes W \cong \mathrm{Hom}(DV, W)$

$\rightsquigarrow$ get new b_n, strictly A_∞-isomorphic to add $\mathcal{A}$

use:

$$\underbrace{\mathrm{Hom}_k\big(\underbrace{D(s\mathcal{A})^{-n}(j,i)}_{\substack{\text{take as}\\ \text{arrows of}\\ \text{a quiver}}},\ \mathrm{Hom}_k(X(i), Y(j))\big)}_{\text{representation of a quiver}}$$

(dual to add $\mathcal{A}^n(X,Y) \ni \mathrm{Hom}_k(X(i), Y(j)) \underset{k}{\otimes} \mathcal{A}^n(j,i)$
+ shift s)

IV. Differential graded algebras

bar construction: $V = s\mathcal{A}$

$\overline{T}V = V \oplus V^{\otimes 2} \oplus \ldots$

with comultiplication $\Delta(v_1, \ldots, v_n) := \sum_{1 \le i \le n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n)$

a graded map $b: \overline{T}V \to V$ of degree one uniquely lifts to a coderivation

$$b: \overline{T}V \to \overline{T}V$$

$$\Delta \circ b = (b \otimes 1 + 1 \otimes b) \circ \Delta$$

m_n for $\mathcal{A}$ $\rightsquigarrow$ b_n for $s\mathcal{A}$ (degree one)

$\rightsquigarrow b: \overline{T}s\mathcal{A} \to s\mathcal{A} \rightsquigarrow b: \overline{T}s\mathcal{A} \to \overline{T}s\mathcal{A}$

$\Rightarrow$ $m_n: \mathcal{A}^{\otimes n} \to \mathcal{A}$ define an A_∞-structure

$\Leftrightarrow b: \overline{T}s\mathcal{A} \to \overline{T}s\mathcal{A}$ satisfies $b^2 = 0$

V. Boxes

$\mathcal{A} = \mathrm{End}^{\infty}_{A}(\Delta, \Delta)$ – actually, degrees 0, 1, 2 suffice

$\rightsquigarrow$

tw $\mathcal{A}$, new A_∞-category, MC-equation

$\rightsquigarrow$

∃ differential graded category

write in terms of conv

representation is in $\mathcal{F}(\Delta)$

$\Leftrightarrow$ it satisfies MC-equation

$\Leftrightarrow$ it composes to zero with d

$\rightsquigarrow$ factoring out d we obtain a finite-dimensional gadget

$$\boxed{\mathcal{F}(\Delta) \simeq \mathfrak{B}\text{-rep}}$$

where $\mathfrak{B} = (B, W, \mu, \varepsilon)$

B directed (qh with Δ's simple)

W B-bimodule

$\mu : W \to W \otimes_B W$ B-coassociative comultiplication

$\varepsilon : W \to B$ surjective B-bimodule morphism, counit for μ

$\mathfrak{B}$ is a box (= bocs, bimodule of coalgebra structure)

directed

projective kernel $0 \to \overline{W} \to W \xrightarrow{\varepsilon} B \to 0$

free

normal

In general: boxes $\overset{1:1}{\longleftrightarrow}$ differential tensor algebras

[R. Bautista, L. Salmerón + R. Zuazua,

Differential tensor algebras and their module categories

LMS Lecture Note Series 362, Cambridge University Press, 2009]

(see: W. Crawley-Boevey, Matrix problems and Drozd's theorem)

VI. Burt-Butler algebras and (co-) induced modules

$\mathfrak{B} = (B, W, \mu, \varepsilon)$ box

a representation of $\mathfrak{B}$ is a representation $X: B \to k\text{-mod}$ of B

given two representations X and Y of $\mathfrak{B}$, the space of morphisms is $\mathfrak{B}(X,Y) := \operatorname{Hom}_B(W \otimes_B X, Y)$

with composition of $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$

$$W \otimes X \xrightarrow{\mu \otimes 1} W \otimes W \otimes X \xrightarrow{1 \otimes f} W \otimes Y \xrightarrow{g} Z$$

and unit element $W \otimes X \xrightarrow{\varepsilon \otimes 1} B \otimes_B X \xrightarrow{\sim} X$

B is a $\mathfrak{B}$-module (left and right) and as such has endomorphism rings (containing B)

the left Burt-Butler algebra of $\mathfrak{B}$ is

$$\operatorname{End}_{\mathfrak{B}}(B_B) =: L$$

the right Burt-Butler algebra of $\mathfrak{B}$ is

$$\operatorname{End}_{\mathfrak{B}}({}_B B) =: R$$

the B-bimodule structure on W extends to

$${}_L W_R$$

and there are equivalences

$$\begin{tikzcd}
 & \mathfrak{B}\text{-representations} \ni {}_B X \arrow[dl, "\operatorname{Hom}_B(L,X)"', "\wr"] \arrow[dr, "R \otimes_B X", "\sim"'] & \\
\mathrm{CoInd} = \{\operatorname{Hom}(L,X)\} \underset{\text{full}}{\subset} L\text{-modules} & & \mathrm{Ind} = \{R \otimes_B X\} \underset{\text{full}}{\subset} R\text{-modules}
\end{tikzcd}$$

Example $\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$

$\mathcal{F}(\Delta)$

$$\begin{tikzcd} & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \arrow[dr] & \\ \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] & & 1 \arrow[ll, bend left] \end{tikzcd}$$

A_∞

$$\begin{tikzcd} 1 \arrow[r, bend left, "1"] \arrow[r, bend right, "0"'] & 2 \end{tikzcd}$$

B $\quad 1 \longrightarrow 2 \qquad 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$

$\mathfrak{B}$

$$\begin{tikzcd} 1 \arrow[loop above, "\omega_1"] \arrow[r, bend left] \arrow[r, bend right, "\varphi"'] & 2 \arrow[loop above, "\omega_2"] \end{tikzcd}$$

coalgebra generated by as proj-B-bimodule

$\omega_1 \overset{\mu}{\longmapsto} \omega_1 \otimes \omega_1$

$\omega_2 \longmapsto \omega_2 \otimes \omega_2$

$\varphi \longmapsto \omega_2 \otimes \varphi + \varphi \otimes \omega_1$

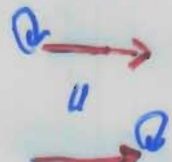
projective B-modules: $k \longrightarrow k$

and $0 \longrightarrow k$

endomorphisms:

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & k \arrow[d, "1"'] & i \\ k \arrow[r, "1"] \arrow[d, "0"'] \arrow[dr, "1"] & k \arrow[d, "0"'] & \varphi \\ 0 \arrow[r, "0"] \arrow[d, "0"'] & k \arrow[d, "1"'] & i \\ k \arrow[r, "1"] & k & \end{tikzcd}$$

relation:



$\Rightarrow$ commutative diagram as for quiver representations

endomorphism algebra

$$\begin{tikzcd} 1 \arrow[r, bend left, "\varphi"] & 2 \arrow[l, bend left, "i"] \end{tikzcd} \Big/ \varphi \circ i = 0$$

$\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$

Summary of the six sides of the box

Δ standard system $\rightsquigarrow$ $\mathcal{F}(\Delta) \subset A$-mod, A f.d.
$\Delta(1) \oplus \ldots \oplus \Delta(n)$

$\downarrow$ } A_∞-structure on $\mathrm{End}^*_A(\Delta,\Delta)$

twisted stalks

copies of $\Delta(i)$'s, Ext^1-elements

$\downarrow$ } + MC equation

new A_∞-structure, conv ($\rightsquigarrow$ quiver)

$\downarrow$ } differential tensor algebra, "factor out MC"

box $\mathfrak{B} = (B, W, \mu, \varepsilon)$

$\downarrow$ } Burt – Butler

$\mathcal{I}nd \subset R$-mod, $\mathcal{C}o\mathcal{I}nd \subset L$-mod

$$\boxed{\begin{array}{l} A \rightsquigarrow \text{directed algebra } B \\ \qquad\quad \text{algebras } L, R \end{array}}$$

Note: all assumptions on Δ have been used

e.g. Ext^* finite dimensional

δ triangular

idempotents split

strictly unital A_∞-structure

...

(for an early example, see: T. Brüstle, K., V. Mazorchuk, The coinvariant algebra and representation types of blocks of category $\mathcal{O}$ Bulletin LMS, 2001)

2. Open the box

A qh, A-mod $> \mathcal{F}(\Delta) \rightsquigarrow B, L, R$

Bautista+Kleiner, Burt+Butler, 1991:

almost split sequences for representations of boxes

"$\Rightarrow$" Ringel's result (1991): $\mathcal{F}(\Delta)$ has almost split sequences

characterisation of quasi-hereditary algebras up to Morita equivalence

an algebra A is quasi-hereditary $\Leftrightarrow$ A is Morita equivalent to $L(\mathfrak{B})$ for some directed box $\mathfrak{B}$

$\Leftrightarrow$ A is Morita equivalent to $R(\mathfrak{B}')$ for some directed box $\mathfrak{B}'$

(wrong when dropping "up to Morita equivalence")

in general $L(\mathfrak{B})$ and $R(\mathfrak{B})$ are not Morita equivalent but:

$$L(\mathfrak{B}) \underset{\text{Morita}}{\sim} R(\underset{\uparrow \text{ Ringel dual}}{R(\mathfrak{B})}) \qquad [\text{see BB}]$$

$L(\mathfrak{B})$ (and $R(\mathfrak{B})$) is qh, contains B as a subalgebra

- B is directed
- ${}_B L(\mathfrak{B})$ (and $R(\mathfrak{B})_B$) is B-projective
- (co-)induction sends simple modules to (co-)standard modules

analogue of Poincaré-Birkhoff-Witt theorem

B is an exact Borel subalgebra of $R(\mathfrak{B})$ [K, 1995]

(have been constructed for blocks of $\mathcal{O}$ – K, 1995

Frobenius kernels – Parshall+Scott+J.P. Wang,

various qh algebras – C.C. Xi, B.M. Deng, – 2000

certain ∞-dim qh algebras – Mazorchuk+Miemietz)

⇒ up to Morita equivalence exact Borel subalgebras do exist for all qh algebras

qh $A \supset B$ exact Borel subalgebra

$\downarrow$

$\mathfrak{B} = (B, W, \mu, \varepsilon)$ directed box

$$\begin{tikzcd} \mathcal{F}(\Delta) & B\text{-mod} \arrow[l, "A\otimes_B -"'] \\ & \mathfrak{B}\text{-rep} \arrow[ul, "\sim", "A\otimes_B -"'] \end{tikzcd}$$

B-mod: not an equivalence, does not preserve representation type

$\mathfrak{B}$-rep: equivalence

$\uparrow$

tame/wild dichotomy