COSUPPORT AND COLOCALIZING SUBCATEGORIES OF MODULES AND COMPLEXES

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Support and cosupport provide a link between

REPRESENTATION THEORY AND GEOMETRY.

I discuss two papers of Amnon Neeman involving these concepts:


Applications in representation theory of finite groups follow at the end.

All this is part of a joint project with D. Benson and S. Iyengar.
Here is the setup:

- $R$ = a commutative noetherian ring
- $\text{Mod } R$ = the category of $R$-modules
- $D(R)$ = the (unbounded) derived category of $\text{Mod } R$
- $\text{Spec } R$ = the set of prime ideals of $R$

$D(R)$ is a triangulated category with set-indexed (co)products.
A triangulated subcategory $C \subseteq D(R)$ is called

- **localizing** if $C$ is closed under taking all coproducts,
- **colocalizing** if $C$ is closed under taking all products.

For any class $S \subseteq D(R)$ write:

$$\text{Loc}(S) = \text{the smallest localizing subcategory containing } S$$
$$\text{Coloc}(S) = \text{the smallest colocalizing subcategory containing } S$$
Theorem (Neeman, 1992)

The assignment

\[ \text{Spec } R \supseteq \mathcal{U} \mapsto \text{Loc}(\{k(p) \mid p \in \mathcal{U}\}) \subseteq D(R) \]

induces a bijection between

- the collection of subsets of \( \text{Spec } R \), and
- the collection of localizing subcategories of \( D(R) \).

Notation: \( k(p) = \text{the residue field } R_p/p_p \)
The assignment

\[ \text{Spec } R \supseteq U \mapsto \text{Coloc}(\{k(p) \mid p \in U\}) \subseteq D(R) \]

induces a bijection between

- the collection of subsets of \( \text{Spec } R \), and
- the collection of colocalizing subcategories of \( D(R) \).

This is surprising because products tend to be complicated!

How are the results from '92 and '09 related to each other?

Is there a common proof?
A consequence / reformulation

For $C \subseteq D(R)$ write:

$C^\perp = \{ X \in D(R) \mid \text{Hom}_{D(R)}(C, X) = 0 \text{ for all } C \in C \}$

$\perp C = \{ X \in D(R) \mid \text{Hom}_{D(R)}(X, C) = 0 \text{ for all } C \in C \}$

- If $C$ is localizing, then $C^\perp$ is colocalizing.
- If $C$ is colocalizing, then $\perp C$ is localizing.
- If $C$ is localizing, then $\perp (C^\perp) = C$ [Neeman 1992].

**Corollary (Neeman, 2009)**

The assignment $C \mapsto C^\perp$ induces a bijection between

- the collection of localizing subcategories of $D(R)$, and
- the collection of colocalizing subcategories of $D(R)$. 
**The support of a complex**

**Definition (Foxby, 1979)**

For $X \in D(R)$ define the support

$$\text{supp } X = \{ p \in \text{Spec } R \mid X \otimes_R^L k(p) \neq 0 \}.$$ 

Some examples:
- If $X \in D^b(\text{mod } R)$, then
  $$\text{supp } X = \{ p \in \text{Spec } R \mid X_p \neq 0 \} = \bigcup_{n \in \mathbb{Z}} \text{supp } H^n(X).$$
- Let $p \in \text{Spec } R$. Then $\text{supp } E(R/p) = \text{supp } k(p) = \{ p \}$.

**Corollary (Neeman, 1992)**

For $X, Y \in D(R)$ we have

$$\text{supp } X \subseteq \text{supp } Y \iff \text{Loc}(X) \subseteq \text{Loc}(Y).$$
**The cosupport of a complex**

**Definition**

For $X \in D(R)$ define the **cosupport**

$$\text{cosupp } X = \{ p \in \text{Spec } R \mid R\text{Hom}_R(k(p), X) \neq 0 \}.$$  

This seems hard to compute, even for ‘simple’ objects:

- Let $R = \mathbb{Z}$. Then $\text{cosupp } X = \text{supp } X$ for $X \in D^b(\text{mod } R)$.
- Let $(R, m)$ be complete local. Then $\text{cosupp } R = \{ m \}$.

**Proposition**

*For a complex $X$ in $D(R)$ we have*

$$\text{Max}(\text{supp } X) = \text{Max}(\text{cosupp } X).$$

**Notation:** $\text{Max} \mathcal{U} = \{ p \in \mathcal{U} \mid p \subseteq q \in \mathcal{U} \implies p = q \}$. 

Four fundamental functors $\text{Mod } R \rightarrow \text{Mod } R$:

- **localization**  \[ M \rightarrow M \otimes_R R_p \]
- **colocalization**  \[ \text{Hom}_R(R_p, M) \rightarrow M \]
- **torsion**  \[ \Gamma_a M = \lim_{\leftarrow} \text{Hom}(R/a^n, M) \rightarrow M \]
- **completion**  \[ M \rightarrow \Lambda_a M = \lim_{\to} M \otimes_R R/a^n \]

Their derived functors $D(R) \rightarrow D(R)$:

- **localization**  \[ X \rightarrow X \otimes^L_R R_p \]
- **colocalization**  \[ \text{RHom}_R(R_p, X) \rightarrow X \]
- **local cohomology**  \[ \text{R} \Gamma_a X \rightarrow X \text{ [Grothendieck, 1967]} \]
- **local homology**  \[ X \rightarrow \text{L} \Lambda_a X \text{ [Greenlees–May, 1992]} \]

Note:

- The functor $\text{RHom}_R(R_p, -)$ is a right adjoint of $- \otimes^L_R R_p$.
- The functor $\text{L} \Lambda_a$ is a right adjoint of $\text{R} \Gamma_a$. 
**Local (co)homology**

**Definition**

Fix $p \in \text{Spec } R$ and define (by abuse of notation):

- **local cohomology** $\Gamma_p = \mathbf{R}\Gamma_p(- \otimes^L_R R_p),$
- **local homology** $\Lambda_p = \mathbf{R}\text{Hom}_R(R_p, L\Lambda_p -).$

These are idempotent functors $D(R) \to D(R)$, and $\Lambda_p$ is a right adjoint of $\Gamma_p$.

We consider their essential images:

- $\text{Im } \Gamma_p = \text{local cohomology objects}$ (a localizing subcategory)
- $\text{Im } \Lambda_p = \text{local homology objects}$ (a colocalizing subcategory)

Note: $\Lambda_p$ induces an equivalence $\text{Im } \Gamma_p \sim \text{Im } \Lambda_p$. 
An alternative description of (co)support:

- $\text{supp} \, X = \{ p \in \text{Spec} \, R \mid \Gamma_p X \neq 0 \}$.
- $\text{cosupp} \, X = \{ p \in \text{Spec} \, R \mid \Lambda_p X \neq 0 \}$.

The following are equivalent:

- $H^n(X)$ is $p$-local and $p$-torsion for all $n \in \mathbb{Z}$.
- $\text{supp} \, X \subseteq \{ p \}$.
- $X$ lies in $\text{Im} \, \Gamma_p$.

There seems to be no analogue for $\Lambda_p$. 
**Proposition**

The assignment

\[
D(R) \supseteq C \longleftrightarrow (C \cap \text{Im } \Gamma_p)_{p \in \text{Spec } R}
\]

induces a bijection between

- the collection of localizing subcategories of \( D(R) \), and
- the collection of families \( (C_p)_{p \in \text{Spec } R} \) with each \( C_p \subseteq \text{Im } \Gamma_p \) a localizing subcategory.

Analogously, the assignment

\[
D(R) \supseteq C \longleftrightarrow (C \cap \text{Im } \Lambda_p)_{p \in \text{Spec } R}
\]

classifies the colocalizing subcategories of \( D(R) \).
**Proposition**

Let \( p \in \text{Spec} \, R \).

- \( \text{Im} \, \Gamma_p \) has no proper localizing subcategories.
- \( \text{Im} \, \Lambda_p \) has no proper colocalizing subcategories.

**Proof.**

For each \( 0 \neq X \in \text{Im} \, \Gamma_p \), one shows that

\[
\text{Loc}(X) = \text{Loc}(k(p)) = \text{Im} \, \Gamma_p.
\]

Analogously, \( \text{Coloc}(Y) = \text{Im} \, \Lambda_p \) for each \( 0 \neq Y \in \text{Im} \, \Lambda_p \).

The classifications of [Neeman, 1992] and [Neeman, 2009] are immediate consequences.
A generalization and an application

The above proof allows to generalize Neeman’s results to the derived category of a differential graded algebra $A$ such that

- $A$ is formal, i.e. quasi-isomorphic to its cohomology $H^*(A)$,
- $H^*(A)$ is graded-commutative and noetherian.

An application to the study of modular representations of finite groups goes as follows:

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. We consider modules over the group algebra $kG$ and classify the (co)localizing subcategories of the stable category $\text{StMod } kG$. 
Take as example $G = (\mathbb{Z}/2\mathbb{Z})^r$ and a field $k$ of characteristic 2.

**Group algebra** $kG \cong k[x_1, \ldots, x_r]/(x_1^2, \ldots, x_r^2)$

**Group cohomology** $H^*(G, k) = \text{Ext}^*_k(k, k) \cong k[\xi_1, \ldots, \xi_r]$

$K(\text{Inj } kG) = \text{category of complexes of injective } kG\text{-modules} / \text{htpy.}$

$i_k = \text{an injective resolution of the trivial representation } k$

$\text{End}_k(i_k) = \text{the endomorphism dg algebra of } i_k \text{ (is formal)}$

$$
\begin{align*}
\text{StMod } kG & \xrightarrow{\sim} K_{\text{ac}}(\text{Inj } kG) \hookrightarrow K(\text{Inj } kG) \\
& \xrightarrow{\sim} D(\text{End}_k(i_k)) \xrightarrow{\sim} D(k[\xi_1, \ldots, \xi_r])
\end{align*}
$$

**Corollary**

There are canonical bijections between

- (co)localizing subcategories of StMod $kG$, and
- sets of graded non-maximal prime ideals of $H^*(G, k)$. 