Quiver mutations and derived equivalences

Conference Interplay between Representation Theory and Geometry, dedicated to the 65th birthdays of Professor Claus Michael Ringel and Professor Kyoji Saito

04 May 2010. Dong Yang

1. The quiver mutation

$Q = (Q_0, Q_1, s, t)$ quiver

Q_0: vertices, Q_1: arrows, s: source, t: target

assume: · Q is finite, i.e. $|Q_0| + |Q_1| < \infty$

· Q has no loops ($\cdot\circlearrowleft$), no 2-cycles ($\cdot \rightleftarrows \cdot$)

Defn (Fomin-Zelevinsky) For $i \in Q_0$, the mutation $\mu_i(Q)$ of Q at the vertex i is the quiver obtained from Q in the following 3 steps

step 1: for each subquiver $j \xrightarrow{\beta} i \xrightarrow{\alpha} j'$, add a new arrow $j \xrightarrow{[\alpha\beta]} j'$;

step 2: replace each arrow $i \xrightarrow{\alpha} j'$ by $i \xleftarrow{\alpha^*} j'$,

and replace each arrow $j \xrightarrow{\beta} i$ by $j \xleftarrow{\beta^*} i$;

step 3: ~~delete~~ the arrows in a maximal set of pairwise disjoint 2-cycles.

Examples. a)

$$\begin{tikzcd} & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] & & 3 \end{tikzcd} \quad \overset{\text{step1}}{\rightsquigarrow} \quad \begin{tikzcd} & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] \arrow[rr, "{[\alpha\beta]}"] & & 3 \end{tikzcd} \quad \overset{\text{step2}}{\rightsquigarrow} \quad \begin{tikzcd} & 2 \arrow[dl, "\beta^*"'] & \\ 1 \arrow[rr, "{[\alpha\beta]}"] & & 3 \arrow[ul, "\alpha^*"'] \end{tikzcd}$$

$i = 2$

b)

$$\begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "c"'] & & 3 \arrow[ul, "b"'] \end{tikzcd} \quad \overset{\text{step1}}{\rightsquigarrow} \quad \begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, shift right, "c"'] & & 3 \arrow[ul, "b"'] \arrow[ll, shift right, "{[ab]}"'] \end{tikzcd} \quad \overset{\text{step2}}{\rightsquigarrow} \quad \begin{tikzcd} & 2 \arrow[dr, "b^*"] & \\ 1 \arrow[ur, "a^*"] \arrow[rr, shift right, "c"'] & & 3 \arrow[ll, shift right, "{[ab]}"'] \end{tikzcd} \quad \overset{\text{step3}}{\rightsquigarrow} \quad \begin{tikzcd} & 2 \arrow[dr, "b^*"] & \\ 1 \arrow[ur, "a^*"] & & 3 \end{tikzcd}$$

$i = 2$

Remarks. 1) $\mu_i^2 = id$

2) we will denote by $\tilde{\mu}_i(Q)$ the quiver obtained after step 2.

Aim: categorify the quiver mutation by the derived equivalence of certain differential graded algebras.

2. The Brenner–Butler tilting

Q : quiver, no loops, no 2-cycles.

$i \in Q_0$

<u>Fact</u>: when i is a sink or a source, the quiver mutation μ_i coincides with the Bernstein–Gelfand–Ponomarev reflection.

k: field

$\widehat{kQ}$:= the complete path algebra

for $j \in Q_0$, P_j := the indec. proj. $\widehat{kQ}$-module corresponding to j.

Let $T = \bigoplus_{j \in Q_0} T_j$, where
$$T_j = \begin{cases} P_j & \text{if } j \neq i \\ \text{cone of } \Big(P_i \xrightarrow{(\beta)} \bigoplus_{\substack{\beta : j \to i \\ \text{in } Q}} P_{s(\beta)} \Big) & \text{if } j = i. \end{cases}$$

If i is a sink, T is (quasi-isomorphic to) a module,

the Auslander–Platzeck–Reiten tilting module.

More generally, if i is not source, T is (quasi-isomorphic to) a module,

the Brenner–Butler tilting module,

which, by Happel's theorem, induces a derived equivalence:

$$\mathcal{D}(\widehat{kQ}) \xleftarrow{T \overset{\mathbb{L}}{\otimes}_B ?} \mathcal{D}(B)$$

where $B = \mathrm{End}_{\widehat{kQ}}(T)^{op}$.

- if i is a sink, $B \cong k\mu_i(Q)$.
- if i is not a sink, $\mathrm{gl.dim}\, B = 2$. We can describe B as a quiver with relations. [quiver: obtained from $\mu_i(Q)$ by deleting those α^*'s, for $\alpha : i \to j'$; relations: $[\alpha\beta]\beta^*$]

If i is a source, T is not (quasi-isomorphic to) a module any longer.

In this case, $B = \mathrm{End}_{\widehat{kQ}}(T) = \widehat{kQ'}$,

where Q' is the graded quiver which has the same underlying quiver as Q,

and the degree of an arrow ρ is
$$\begin{cases} 0 & \text{if } s(\rho) \neq i \\ -1 & \text{if } s(\rho) = i \end{cases}$$

<u>Example</u>: $Q = \begin{tikzcd}[column sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3\end{tikzcd}$, $i = 2$, $T = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1 \oplus 3$

$B = k \begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "c"'] & & 3\end{tikzcd} \Big/ (ca)$.

Clearly, if we replace the relation $ca: 2\to 3$ ~~ca~~ by an arrow $b: 3 \to 2$,

then we get the quiver, $\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "c"'] & & 3 \arrow[ul, "b"']\end{tikzcd} = \mu_2(Q)$.

The relation ca is, to some extent, encoded in the cycle bca.

Namely, we get a pair $\left(\begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "a"'] & \\ \cdot \arrow[rr, "c"'] & & \cdot \arrow[ul, "b"']\end{tikzcd} , \ bca \right)$, which is a quiver with potential.

3. Quivers with potential

A quiver with potential is a pair (Q, W),

where Q is a finite quiver,

and W is a potential, i.e. $W = \sum_{\substack{c:\text{cycle} \\ \ell(c) \geq 2}} \lambda_c c \in \widehat{kQ}$. [appeared in the talk of K. Hori on the Landau-Ginzburg model side]

For what we will consider, it doesn't matter if we cyclically permute a cycle. In this way, W can be considered as an element of $HH_0(\widehat{kQ}) = \widehat{kQ} \big/ [\widehat{kQ}, \widehat{kQ}]$.

$i \in Q_0$ + suitable assumptions.

<u>Defn</u> (Derksen-Weyman-Zelevinsk) ~~...~~

$$\mu_i(Q, W) := \text{the 2-reduction of } \widetilde{\mu}_i(Q,W) = (\widetilde{\mu}_i(Q), \widetilde{W}),$$

where $\widetilde{W} = [W] + \Delta$,

$[W] = W|_{\text{replace all } \alpha\beta \text{ by } [\alpha\beta]}$.

$\Delta = \sum_{\beta: j \to i} \sum_{\alpha: i \to j'} [\alpha\beta] \beta^* \alpha^*$.

<u>Examples</u>: a) $\begin{tikzcd}[column sep=small] & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] & & 3\end{tikzcd}$ $\xrightarrow{\widetilde{\mu}_2}$ $\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "\beta^*"'] & \\ 1 \arrow[rr, "{[\alpha\beta]}"'] & & 3 \arrow[ul, "\alpha^*"']\end{tikzcd}$

$i = 2$
$W = 0$
$\widetilde{W} = [\alpha\beta]\beta^*\alpha^*$

b) $\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "c"'] & & 3 \arrow[ul, "b"']\end{tikzcd}$ $\xrightarrow{\widetilde{\mu}_2}$ $\begin{tikzcd}[column sep=small] & 2 \arrow[dr, "b^*"] & \\ 1 \arrow[ur, "a^*"] \arrow[rr, shift left, "{[ab]}"] & & 3 \arrow[ll, shift left, "c"]\end{tikzcd}$ $\xrightarrow{\text{2-reduction}}$ $\begin{tikzcd}[column sep=small] & 2 \arrow[dr, "b^*"] & \\ 1 \arrow[ur, "a^*"] & & 3\end{tikzcd}$

$i = 2$
$W = cab$
$\widetilde{W} = c[ab] + [ab]b^*a^* = [ab](c + b^*a^*)$

4. Ginzburg dg algebras and the main theorem

(Q,W) : quiver with potential

<u>Defn</u> (Ginzburg) $\Gamma(Q,W) := (\widehat{k\tilde{Q}}, d)$, dg algebra

where $\tilde{Q}$ is a graded quiver with $\tilde{Q}_0 = Q_0$

whose arrows consist of three kinds

- those arrows of Q, put in degree 0
- an arrow $\rho^* : i \to j$ for each $\rho : j \to i$ in Q, put in degree -1
- a loop $t_i : i \to i$ for each $i \in Q_0$, put in degree -2

and $d : \widehat{k\tilde{Q}} \longrightarrow \widehat{k\tilde{Q}}$ is the unique k-linear transformation such that

- d is continuous
- d satisfies the graded Leibniz rule, i.e.

  $d(ab) = d(a)b + (-1)^{|a|} a\, d(b)$, for a, b homogeneous

  $|a|$ = the degree of a

- d takes the following values on arrows of $\tilde{Q}_1$
  - $d(\rho) = 0$ for $\rho \in Q_1$
  - $d(\rho^*) = \partial_\rho W$ for $\rho \in Q_1$, where $\partial_\rho : HH_0(\widehat{kQ}) \to \widehat{kQ}$ is the cyclic derivative. (It is a noncomm. version of the differential, roughly speaking, it removes ρ from a cycle.)
  - $d(t_i) = e_i \left(\sum_{\rho \in Q_1} (\rho\rho^* - \rho^*\rho) \right) e_i$, for $i \in Q_0$,

    where e_i is the trivial path at i.

<u>Example</u>. $Q = $
$$\begin{tikzcd} & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] & & 3 \end{tikzcd}$$

$W = 0$

$\tilde{Q} = $
$$\begin{tikzcd} & 2 \arrow[loop above, "t_2"] \arrow[dl, shift right, "\beta^*"'] \arrow[dr, shift left, "\alpha"] & \\ 1 \arrow[loop below, "t_1"] \arrow[ur, shift right, "\beta"'] & & 3 \arrow[loop right, "t_3"] \arrow[ul, shift left, "\alpha^*"] \end{tikzcd}$$

$d(\alpha) = 0$, $d(\beta) = 0$, $d(\alpha^*) = 0$, $d(\beta^*) = 0$

$\sum_{\rho \in Q_1} (\rho\rho^* - \rho^*\rho) = \alpha\alpha^* - \alpha^*\alpha + \beta\beta^* - \beta^*\beta$

$d(t_1) = -\beta^*\beta$, $d(t_2) = -\alpha^*\alpha + \beta\beta^*$, $d(t_3) = \alpha\alpha^*$

$\mathcal{D}(\Gamma(Q,W))$: the derived category of the dg algebra $\Gamma(Q,W)$

Rk. a) $\mathcal{D}(\Gamma(Q,W))$ has a triangulated subcategory $\mathcal{D}_{fd}(\Gamma(Q,W))$, which is 3-Calabi-Yau as a triangulated category.

Kontsevich-Soibelmann defined an A_∞-algebra $\mathcal{A}$.

By Koszul duality we have a triangle equivalence

$$\begin{array}{c} \mathcal{D}_{fd}(\Gamma(Q,W)) \\ \downarrow \simeq \text{ Koszul duality.} \\ \mathrm{Tr}\mathcal{A} = H^0\mathrm{Tw}(\mathcal{A}) \end{array}$$

(cf. talks of H. Kajiura, S. König for defn of $\mathrm{Tr}\mathcal{A}$)

(cf. the talk of K. Hori for a similar picture)

b) $H^0\Gamma(Q,W) = \widehat{kQ} / (\partial_\rho W, \rho \in Q_1)$ is a noncommutative version of the Jacobi algebra mentioned in the talk of K. Hori.

Main Thm (Keller-Y). $i \in Q_0$

$$\mathcal{D}(\Gamma(Q,W)) \simeq \mathcal{D}(\Gamma(\mu_i(Q,W))) \text{ as triangulated categories.}$$

In the paper of B. Keller and myself, we gave a direct proof. Namely, we constructed a dg bimodule and showed that the associated derived tensor functor is a triangle equivalence.

In the next section I present a proof using Calabi-Yau completions, due to Keller.

5. Calabi-Yau completions and the proof.

5. Calabi-Yau completions and the proof

A: homologically smooth topological dg algebra, i.e. $A \in \mathrm{per}(A^e) = \mathrm{thick}(A^e) \subseteq \mathcal{D}(A^e)$
$A^e = A^{op} \otimes_k A$

$\Theta := R\mathrm{Hom}_{A^e}(A, A^e)[2]$

$c \in HH_0(A)$

Defn (Keller) The deformed 3-CY completion of A is the tensor algebra $\overbrace{\hat{T}_A \Theta}^{\Pi_3(A,c)} = A \oplus \Theta \oplus \Theta \hat{\otimes}_A \Theta \oplus \cdots$
whose differential is deformed by c.

Thm (Keller)

a) Let (Q, W) be a quiver with potential.

Then $\Gamma(Q, W) = \Pi_3(\widehat{kQ}, W)$

b) Let $B = \widehat{kQ}/I$ be an ordinary algebra considered as a dg algebra concentrated in degree 0. Assume gl.dim $B = 2$ and assume $I = \overline{\langle r_1, \cdots, r_m \rangle}$ ($r_1, \cdots, r_m$ a set of minimal relations). Let $c \in HH_0(B)$, lifted to a potential W_0 of Q. Then we have

$$\Pi_3(B, c) \underset{q.is.}{\simeq} \Gamma(Q', W'),$$

where $Q' = Q \amalg \{\rho_i : t(r_i) \to s(r_i) \mid 1 \leq i \leq m\}$

$$W' = W_0 + W_1, \quad W_1 = \sum_{i=1}^{m} \rho_i r_i$$

c) a standard derived equivalence $\mathcal{D}(A) \simeq \mathcal{D}(A')$ sending $c \in HH_0(A)$ to $c' \in HH_0(A')$ induces a derived equivalence

$$\mathcal{D}(\Pi_3(A, c)) \simeq \mathcal{D}(\Pi_3(A', c')).$$

The proof of the Main Thm by Keller:

(Q, W) quiver with potential

$i \in Q_0$ a vertex

Recall from section 2 that $\exists$ standard derived equivalence

$$\mathcal{D}(\widehat{kQ}) \xrightarrow{\sim} \mathcal{D}(B)$$
$$W \longmapsto [W]$$

By part c) of the preceding theorem, we have

$$\mathcal{D}(\Pi_3(\widehat{kQ}, W)) \xrightarrow{\sim} \mathcal{D}(\Pi_3(B, [W]))$$

LHS $=$ $\mathcal{D}(\Gamma(Q,W))$ by part a)

3 cases for RHS:

case 1: i is a sink. In this case, $B = k\widehat{\mu_i(Q)}$, and $[W] = W$,

and $\mu_i(Q,W) = (\mu_i(Q), W)$

So RHS $=$ $\mathcal{D}(\Gamma(\mu_i(Q,W)))$ by part a)

case 2: i is not a sink, not a source.

In this case, RHS $\cong \mathcal{D}(\Gamma(Q', W'))$ by part b).

It turns out that $(Q', W') = \tilde{\mu}_i(Q,W)$.

The 2-reduction yields $\mathcal{D}(\Gamma(\tilde{\mu}_i(Q,W))) \cong \mathcal{D}(\Gamma(\mu_i(Q,W)))$

So RHS $\cong \mathcal{D}(\Gamma(\mu_i(Q,W)))$

case 3: i is a source. In this case, $B = \widehat{kQ'}$, $[W] = W$, cf. section 2.

It is not difficult to see that

$$\Pi_3(B, W) = \Gamma(\mu_i(Q), W) = \Gamma(\mu_i(Q,W)).$$

So RHS $=$ $\mathcal{D}(\Gamma(\mu_i(Q,W)))$.

In all cases, we have $\mathcal{D}(\Gamma(Q,W)) \xrightarrow{\sim} \mathcal{D}(\mu_i(\Gamma(Q,W)))$,

as desired.

$\square$

Example: $Q = \begin{tikzcd}[column sep=small] & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] & & 3 \end{tikzcd}$, $W=0$, $i=2$

$$kQ \underset{\text{der}}{\sim} k\left(\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] \\ 1 \arrow[r, "c"'] & 3 \end{tikzcd}\right) / ca$$

$\wr$ 3-CY completion

$$\Pi_3(kQ, 0) \underset{\text{der}}{\sim} \Pi_3\left(k\left(\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] \\ 1 \arrow[r, "c"'] & 3 \end{tikzcd}\right) / ca, \ 0 \right)$$

$$\| \qquad\qquad\qquad\qquad \wr \text{ q.is.}$$

$$\Gamma(Q, 0) \qquad\qquad \Gamma\left(\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "c"'] & & 3 \arrow[ul, "b"'] \end{tikzcd}, \ cab \right)$$

$$\|$$

$$\Gamma(\mu_2(Q, 0))$$

$\square$