Minimal presentation for fundamental group of Complement of hyperplane arrangement

§. Minimal CW complex

X : a finite CW complex.

<u>Def</u> X is <u>minimal</u>

$\underset{def}{\Longleftrightarrow}$ # of k-dim cells $= b_k(X)$

<u>Rem</u> In general : # of k-dim cells $\geqq b_k(X)$

<u>Thm</u> (Dimca-Papadima, Randell)

$\mathcal{A} = \{H_1 \cdots H_n\}$ $H_i \subset \mathbb{C}^\ell$ affine hyperplane

$M := M(\mathcal{A}) := \mathbb{C}^\ell \setminus \bigcup H_i$, then

M is homotopy equivalent to a minimal CW complex.

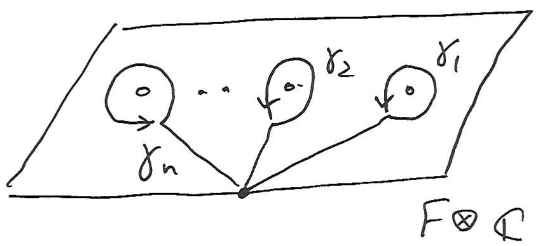
Considering the case $\ell = 2$

$\rightsquigarrow$ minimal presentation for $\pi_1(M)$

$\left\{\begin{array}{l} \#\text{ of generators} = b_1 \\ \#\text{ of relations} = b_2 \end{array}\right.$

$H_i \cap F \leftrightarrow$ generator

$ch_F(\mathcal{A}) \leftrightarrow$ relation



<u>Prop</u> $\mathcal{A} = \{H_1, \cdots, H_n\}$

$$\# ch_F(\mathcal{A}) = b_2(M)$$

(proof)

$$\begin{array}{ccc} \#\text{ of chambers} & \overset{\text{Zaslawski}}{=} & \underset{\displaystyle 1}{\underset{\shortparallel}{b_0}} + \underset{\displaystyle n}{\underset{\shortparallel}{b_1}} + b_2 \\ \shortparallel & & \\ n+1+\#ch_F(\mathcal{A}). & & \end{array}$$

∎

<u>Thm</u> For $C \in ch_F(\mathcal{A})$,

$\exists!$ (unique up to homotopy) continuous map

$$\sigma_C : (D^2, \partial D^2) \longrightarrow (M, M \cap F)$$

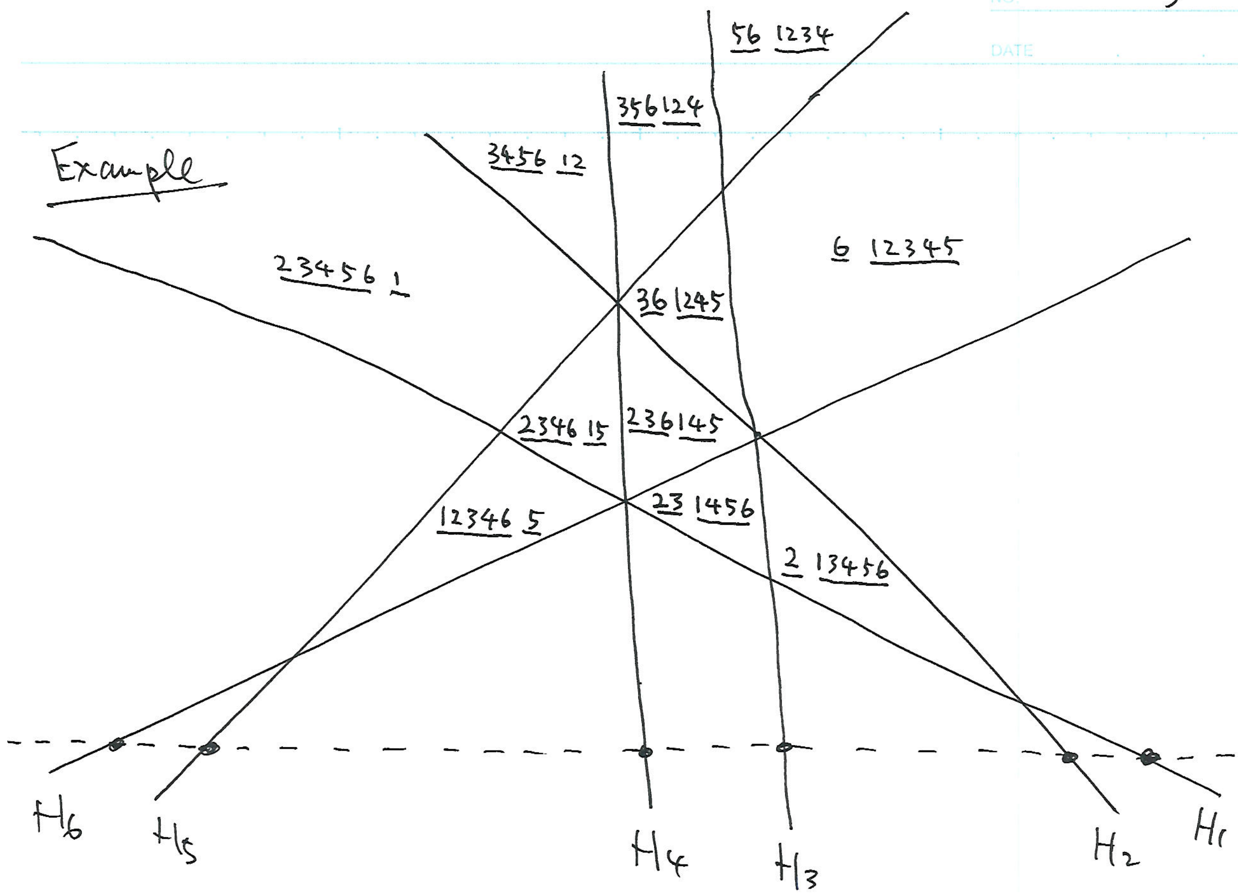
s.t.

(i) $\sigma_C(D^2) \pitchfork C$: transversal

(ii) $\sigma_C(D^2) \cap C = \{pt\}$

(iii) $\sigma_C(D^2) \cap C' = \phi$ for $C' \in ch_F(\mathcal{A}) \setminus \{C\}$

Example



$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \;\middle|\; \begin{aligned} \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 &= 213456 = 231456 \\ &= 123465 = 234615 \\ &= 236145 = 361245 \\ &= 356124 = 234561 \\ &= 345612 = 561234 = 612345 \end{aligned} \right\rangle$$